Math 491, Problem Set #11 (Solutions)

1. *Let c_n be the number of domino tilings of a 3-by-$2n$ cylinder, obtained by gluing together the left and right sides (of length 3) of a 3-by-$2n$ rectangle. Express the generating function $c_1x + c_2x^2 + c_3x^3 + \ldots$ as a rational function of x.*

   Where two squares meet side by side, record a 0 if they are covered by a single (horizontal) domino and a 1 otherwise. Then there are eight possible patterns of 0's and 1's as one reads down, which we treat as numbers between 0 through 7 via binary encoding. Then the domino tilings of the 3-by-$2n$ cylinder correspond to circular words of length $2n$ in the symbols 0 through 7, where the allowed transitions are summarized in the following matrix (with rows and columns indexed consecutively from 0 through 7):

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

   That is, in Maple-ese,

```
M := matrix(8,8,[
0,1,0,0,0,0,0,0,
0,0,0,0,0,0,1,0,
0,0,0,0,0,1,0,0,
0,0,0,0,1,0,0,1,
0,0,0,1,0,0,0,0,
0,0,1,0,0,0,0,0,
0,1,0,0,0,0,0,1,
1,1,0,0,1,0,0,0]);
```

Maple, in response to the command `charpoly(M,t)`, tells us that the characteristic polynomial of this matrix is $\det(tI - M) = t^8 - 3t^6 - 2t^5 + 2t^4 + 4t^3 + t^2 - 2t - 1$, so the twisted version is $Q(x) = \det(I - xM) = 1 - 3x^2 - 2x^3 + 2x^4 + 4x^5 + x^6 - 2x^7 - x^8$, which factors as $(1-x)^2(1+x)^2(1-x-x^2)(1+x+x^2)$. Therefore $\frac{-xQ'(x)}{Q(x)} = \frac{2x(1-x)}{(1-x)^2} - \frac{2x(1+x)}{(1+x)^2} + \frac{x+2x^2}{1-x-x^2} + \frac{-x-2x^2}{1+x+x^2} = \frac{6x^2+6x^3-2x^4-14x^5-8x^6}{(1-x)(1+x)(1-x-x^2)(1+x+x^2)}$.