

Math 491, Problem Set #14: Solutions

1. Consider the subset of the square grid bounded by the vertices $(0, 0)$, $(m, 0)$, $(0, n)$, and (m, n) , and let q be a formal indeterminate. Let the weight of the horizontal grid-edge joining (i, j) and $(i + 1, j)$ be q^j (for all $0 \leq i \leq m - 1$ and $0 \leq j \leq n$), and let each vertical grid-edge have weight 1. Define the weight of a lattice path of length $m + n$ from $(0, 0)$ to (m, n) to be the product of the weights of all its constituent edges. Let $P(m, n)$ be the sum of the weights of all the lattice paths of length $m + n$ from $(0, 0)$ to (m, n) , a polynomial in q . (Note that putting $q = 1$ turns $P(m, n)$ into the number of lattice paths of length $m + n$ from $(0, 0)$ to (m, n) , which is the binomial coefficient $\frac{(m+n)!}{m!n!}$.)

- (a) Give a formula for $P(1, n)$ and for the generating function

$$\sum_{n \geq 0} P(1, n)x^n.$$

The $n + 1$ paths from $(0, 0)$ to $(1, n)$ have weight $1, q, q^2, \dots, q^n$, so $P(1, n) = 1 + q + q^2 + \dots + q^n = (1 - q^{n+1})/(1 - q)$ and

$$\begin{aligned} \sum_{n \geq 0} P(1, n)x^n &= \sum_{n \geq 0} \frac{x^n - q^{n+1}x^n}{1 - q} \\ &= \sum_{n \geq 0} \left(\frac{x^n}{1 - q} - \frac{q(qx)^n}{1 - q} \right) \\ &= \frac{1}{(1 - q)(1 - x)} - \frac{q}{(1 - q)(1 - qx)} \\ &= \frac{1}{(1 - x)(1 - qx)} \end{aligned}$$

- (b) Find (and justify) a recurrence relation relating the polynomials $P(m, n)$, $P(m - 1, n)$, and $P(m, n - 1)$ that generalizes the Pascal triangle relation for binomial coefficients.

A path p from $(0, 0)$ to (m, n) passes through either $(1, 0)$ or $(0, 1)$, but not both.

In the first case, let p' be the path from 0 to $(m-1, n)$ obtained from p by snipping out the step from $(0, 0)$ to $(1, 0)$ and sliding the rest of the path one step to the left. In this case the weight of p' equals the weight of p .

In the second case, let p' be the path from 0 to $(m, n-1)$ obtained from p by snipping out the step from $(0, 0)$ to $(0, 1)$ and sliding the rest of the path one step downward. In this case the weight of p' equals the weight of p times q^m (since each of the m horizontal edges of p has weight equal to q times the weight of the corresponding horizontal edge of p').

Combining, we find that

$$P(m, n) = P(m-1, n) + q^m P(m, n-1).$$

Indeed, we can check this against (a), using the trivial case $P(0, n) = 1$: the relation $P(1, n) = P(0, n) + q^1 P(1, n-1)$ then amounts to $1 + q + q^2 + \dots + q^n = 1 + q(1 + q + \dots + q^{n-1})$, which is true.

- (c) Let $F_m(x)$ denote $\sum_{n \geq 0} P(m, n)x^n$. Use your answer from (b) to give a formula for $F_m(x)$ in terms of $F_{m-1}(x)$, and from this derive a non-recursive formula for $F_m(x)$.

Multiply the inset equation by x^n and sum over all $n \geq 1$ (noting that the omitted term $P(m, 0)x^0$ is just 1):

$$F_m(x) - 1 = (F_{m-1}(x) - 1) + q^m x F_m(x)$$

This gives $(1 - xq^m)F_m(x) = F_{m-1}(x)$ so that

$$F_m(x) = F_{m-1}(x)/(1 - xq^m).$$

Indeed, using this relation and the base case $F_0(x) = 1/(1-x)$ we get the general formula

$$F_m(x) = \frac{1}{(1-x)(1-xq)\cdots(1-xq^m)}.$$

- (d) Write a computer program to compute the polynomial $P(m, n)$ for any input values m, n .

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readlib(coeftayl);
F := proc(m) local i; product(1/(1-q^i*x),i=0..m); end;
P := proc(m,n) simplify(coeftayl(F(m),x=0,n)); end;
r := proc(m,n)
    simplify(P(m,n)*P(m-1,n-1)/P(m-1,n)/P(m,n-1)); end;

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Note that the function `coeftayl` is a good thing to use when you want just one coefficient from a Taylor expansion; it saves time.

- (e) Compute $P(m,n)/P(m-1,n)$ for various values of $m \geq 1$ and $n \geq 0$ and conjecture a formula for it. Do the same for the ratio $P(m,n)/P(m,n-1)$ with $m \geq 0$ and $n \geq 1$.

A bit of playing shows that the first ratio equals

$$(1 + q + q^2 + \dots + q^{m+n-1}) / (1 + q + q^2 + \dots + q^{m-1})$$

or $(1 - q^{m+n}) / (1 - q^m)$, while the second ratio equals or $(1 - q^{m+n}) / (1 - q^n)$. That is, we conjecture that

$$P(m,n)/P(m-1,n) = (1 - q^{m+n}) / (1 - q^m)$$

for $m \geq 1$ and $n \geq 0$ and

$$P(m,n)/P(m,n-1) = (1 - q^{m+n}) / (1 - q^n)$$

for $m \geq 0$ and $n \geq 1$.

- (f) Use the recurrence relation from (b) to verify your conjectures from (e).

We prove the two conjectures simultaneously by joint induction on m, n . That is, we verify the first formula for its base cases (where $m = 1$ or $n = 0$) and the second formula for its base cases (where $m = 0$ or $n = 1$), and we then verify that both formulas hold for m, n if both formulas are both assumed to hold when m, n are replaced by integers m', n' satisfying $m' + n' < m_n$.

We have $P(0,n) = 1$ and $P(1,n) = (1 - q^{n+1}) / (1 - q)$ for all $n \geq 0$, so $P(1,n)/P(0,n) = (1 - q^{1+n}) / (1 - q^1)$ for all $n \geq 0$ and $P(m,0)/P(m-1,0) = (1 - q^{m+0}) / (1 - q^m)$ for all $m \geq 1$, as claimed. It is also easy to check that $P(m,0) = 1$ and

$P(m, 1) = (1 - q^{m+1})/(1 - q)$ for all $m \geq 0$, so $P(0, n)/P(0, n - 1) = (1 - q^{0+n})/(1 - q^n)$ for all $n \geq 1$ and $P(m, 1)/P(m, 0) = (1 - q^{m+1})/(1 - q^1)$ for all $m \geq 0$, as claimed.

Next we use the induction hypothesis to prove $P(m, n)/P(m - 1, n) = (1 - q^{m+n})/(1 - q^m)$. Rewrite this as $P(m, n) = P(m - 1, n)(1 - q^{m+n})/(1 - q^m)$. We know (from (b)) that $P(m, n) = P(m - 1, n) + q^m P(m, n - 1)$, so the thing we're trying to prove can be rewritten as $P(m - 1, n)(1 - q^{m+n})/(1 - q^m) = P(m - 1, n) + q^m P(m, n - 1)$ or

$$P(m - 1, n)(1 - q^n)/(1 - q^m) = P(m, n - 1).$$

But by the induction hypothesis we have $P(m - 1, n) = P(m - 1, n - 1)(1 - q^{m-1+n})/(1 - q^n)$ and $P(m, n - 1) = P(m - 1, n - 1)(1 - q^{m+n-1})/(1 - q^m)$; dividing the first of these by the second gives us what we need to prove.

Lastly we use the induction hypothesis to prove $P(m, n)/P(m, n - 1) = (1 - q^{m+n})/(1 - q^n)$. Rewrite this as $P(m, n) = P(m, n - 1)(1 - q^{m+n})/(1 - q^n)$. We know (from (b)) that $P(m, n) = P(m - 1, n) + q^m P(m, n - 1)$, so the thing we're trying to prove can be rewritten as $P(m, n - 1)(1 - q^{m+n})/(1 - q^n) = P(m - 1, n) + q^m P(m, n - 1)$ or

$$P(m, n - 1)(1 - q^m)/(1 - q^n) = P(m - 1, n),$$

which we proved in the preceding paragraph.

This completes the proof.

Note that we can conclude as a corollary that $P(m, n) = P(n, m)$ for all m, n (though as we'll see there are easier ways to prove this!).