Math 491, Problem Set #17: Solutions

1. *Let $p(n)$ be the number of unconstrained partitions of $n$ if $n \geq 0$, and 0 othewise, so that*

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)--++\ldots$$

*for all $n > 0$. Use the recurrence for $p(n)$ to compute the last digit of $p(n)$ for every $n$ between 1 and 1000. Make a conjecture about the relationship between the last digit of $n$ and the last digit of $p(n)$; specifically, make a conjecture about which pairs $(n \text{ mod } 10,\ p(n) \text{ mod } 10)$ occur and which don't.*

Here's a Maple program that does this:

```
F := proc(n) option remember; local total, k;
if n=0 then 1; elif n<0 then 0; else total := 0;
k := 1; while k*(3*k+1)/2 <= n do
total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k+1: od:
k := -1; while k*(3*k+1)/2 <= n do
total := total - (-1)^k*F(n-k*(3*k+1)/2): k := k-1: od:
total mod 10; fi: end;
```

We then create a matrix to keep track of how often it happens that $n$ ends with the digit $i$ while $p(n)$ ends with the digit $j$ (for $i, j$ between 0 and 9), and print out its entries:

```
for i from 0 to 9 do for j from 0 to 9 do a[i,j]:=0: od: od:
for n from 1 to 1000 do k: = F(n);
       a[n mod 10, k mod 10] := a[n mod 10, k mod 10] + 1; od:
for i from 0 to 9 do seq(a[i,j],j=0..9) od;
```

This results in the output

```
14, 7, 13, 12, 3, 5, 12, 15, 9, 10
9, 11, 14, 9, 8, 9, 10, 13, 7, 10
3, 14, 12, 14, 10, 8, 8, 12, 6, 13
 8, 9, 12, 9, 8, 17, 5, 13, 9, 10
```

```
49, 0, 0, 0, 0, 51, 0, 0, 0, 0
5, 12, 10, 4, 15, 7, 13, 13, 8, 13
8, 14, 11, 15, 7, 9, 8, 6, 5, 17
9, 10, 6, 9, 16, 10, 9, 9, 12, 10
10, 9, 16, 8, 11, 11, 11, 13, 5, 6
48, 0, 0, 0, 0, 52, 0, 0, 0, 0
```

from which we conjecture that when $n$ ends in 4 or 9, $p(n)$ ends in 0 or 5. That is, if $n$ is 1 less than a multiple of 5, $p(n)$ is a multiple of 5.

This fact was first noticed and proved by Ramanujan. Coincidentally, Prof. Ono spoke about this very result in his Math Club talk yesterday (December 1)!

2. *Let $f(0) = 1$ and recursively define $f(n) = f(n-1) + f(n-3) - f(n-6) - f(n-10) + f(n-15) + f(n-21) - - + +\ldots$ for all $n > 0$, where terms of the form $f(n-k)$ are to be ignored once $k > n$.*

   (a) Since the formal power series $F(q) = \sum_{n\geq 0} f(n)q^n = 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 7q^7 + \ldots$ has constant term 1, we saw in class that it admits a (unique) convergent infinite formal product expansion of the form

   $$(1-q)^{a_1}(1-q^2)^{a_2}(1-q^3)^{a_3}(1-q^4)^{a_4}\cdots$$

   Find $a_1$ through $a_{24}$, and conjecture a general rule.
   The defining recursion for $f(n)$ tells us that $F(q) = 1/(1-q-q^3+q^6+q^{10}-q^{15}-q^{21}+\ldots)$. So we may start from this expression (say using exponents up to 25), take its formal Taylor series, and repeatedly divide or multiply by a suitable expression of the form $(1-q^m)^k$ so as to increase the degree of the first non-constant term in the (continually modified) series.
   More specifically, suppose we start with the command

   ```
   taylor(1/(1-q-q\^3+q\^6+q\^10-q\^15-q\^21),q,25);
   ```

   which returns a power series with constant term 1 and linear term $q$. We can get the coefficient of the linear term to become 0 if we multiply by $1-q$. Entering

   ```
   taylor(%*(1-q),q,25);
   ```

we get a power series with constant term 1, no linear term, no quadratic term, and a cubic term $q^3$. We can get the coefficient of the cubic term to become 0 if we multiply by $1-q^3$. So we enter

```
taylor(%*(1-q^3),q,25);
```

And so on. Proceeding in this fashion, we find that $a_1 = -1$, $a_2 = 0$, $a_3 = -1$, $a_4 = -1$, $a_5 = -1$, $a_6 = 0$, $a_7 = -1$, $a_8 = -1$, etc.; that is, for all $n \leq 24$, $a_n$ is 0 if $n$ is 2 more than a multiple of 4 and is $-1$ otherwise.

(b) *Assuming that your answer from (a) is correct, prove that for a particular set $S$ of positive integers (which you must find!), $f(n)$ equals the number of partitions of $n$ into parts belonging to $S$.*

Since $a_n$ is always either $-1$ or 0, this is simple: $(1-q)^{a_1}(1-q^2)^{a_2}(1-q^3)^{a_3}(1-q^4)^{a_4}\cdots$ is $\prod_{k\in S}(1-q^k)^{-1}$, so $S$ is the set of all positive integers that are either odd or divisible by 4.

(c) *Prove that your conjectures from (a) and (b) are correct, e.g. by using the Jacobi triple product identity*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2}z^{2m}$$

*(which you do not need to prove). An equivalent form of the Jacobi triple product identity is*

$$\prod_{i=1}^{\infty}(1+xq^i)(1+x^{-1}q^{i-1})(1-q^i) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}x^n.$$

We want to equate the sum

$$\ldots + x^9z^{-6} + x^4z^{-4} + x^1z^{-2} + x^0z^0 + x^1z^2 + x^4z^4 + x^9z^6 + \ldots$$

(the right hand side of the Jacobi triple product identity) with the sum

$$\ldots - q^{15} + q^6 - q + 1 - q^3 + q^{10} - q^{21} + \ldots$$

(the $q$-series we are trying to evaluate in product form) by making a suitable choice for $x$ and $z$; we achieve this by setting $x = q^2$ and $z = i\sqrt{q}$ (so that $z^2 = -q$). Then the left hand side of the Jacobi triple product identity becomes $\prod_{i=1}^{\infty}(1-q^{4n})(1-q^{4n-1})(1-q^{4n-3})$. So we have shown that

$$1/F(q) = \prod \frac{1}{1-q^m}$$

where the product is taken over all $m$ that are not 2 more than a multiple of 4. This is equivalent to what was conjectured in part (b).