Math 491, Problem Set #3: Solutions

1. (a) *Consider the sequence* $1, 1, 1, 3, 3, 7, 9, 17, 25, \ldots$ *satisfying the initial conditions* $a_0 = a_1 = a_2 = 1$ *and the recurrence relation* $a_n = 2a_{n-2} + a_{n-3}$. *Write the generating functions* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *as a rational function of* x, *expressed in simplest terms.*

   If we write the recurrence in the form, $1a_n - 0a_{n-1} - 2a_{n-2} - 1a_{n-3} = 0$ we see that the denominator of the generating function will be $1 - 0x - 2x^2 - 1x^3$. Multiplying $1 + x + x^2 + 3x^3 + 3x^4 + 7x^5 + 9x^6 + 17x^7 + 25x^8 + \ldots$ by $1 - 2x^2 - x^3$, we get $A(x) = (1 + x - x^2)/(1 - 2x^2 - x^3)$. If we have access to Maple, we can easily check this answer by giving it the query

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taylor((1+x-x^2)/(1-2*x^2-x^3),x,10);
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   (b) *Find an exact formula for* a_n.

   The command

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convert((1+x-x^2)/(1-2*x^2-x^3),parfrac,x);
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   ("convert $(1 + x - x^2)/(1 - 2x^2 - x^3)$ into partial fractions, to the extent that one can do this without introducing irrational numbers") takes us part of the way there, by telling us that $A(x)$ is $-1/(1 + x)$ plus $(2 - 2x)/(1 - x - x^2)$. (Actually, Maple writes the answer as

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-1/(x+1)+2*(x-1)/(x^2+x-1)
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   and leaves it to us to notice that in this context, it makes more sense to list the coefficients of polynomials from lowest degree to highest rather than the other way around.)

   The coefficient of x^n in $-1/(1+x)$ is $(-1)^{n+1}$, and we can compute the coefficient of x^n in $2(x-1)/(x^2+x-1)$ by the procedure that Wilf uses in his treatment of the Fibonacci numbers; or, we can be clever and hitch a ride on the already-known formula for Fibonacci numbers. If we subtract $-1/(1 + x) = -1 + x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + \ldots$ from $A(x) = 1 + x + x^2 + 3x^3 + 3x^4 + 7x^5 + 9x^6 + 17x^7 + 25x^8 + \ldots$, we get $2 + 0x + 2x^2 + 2x^3 + 4x^4 + 6x^5 +$

$10x^6 + 16x^7 + 26x^8 + \ldots$; if we subtract 2 from this g.f. and divide by $2x^2$, we get $f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^6 + 13x^7 + \ldots$, which Wilf has already analyzed for us: the coefficient of x^n in this g.f. is $(\alpha^{n+1} - \beta^{n+1})/\sqrt{5}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. So the coefficient of x^n in $2 + 0x + 2x^2 + 2x^3 + 4x^4 + 6x^5 + \ldots$ is $2(\alpha^{n-1} - \beta^{n-1})/\sqrt{5}$, and the coefficient of x^n in $A(x)$ is $2(\alpha^{n-1} - \beta^{n-1})/\sqrt{5} - (-1)^n$.

(c) *Why did I use the recurrence $a_n = 2a_{n-2} + a_{n-3}$ for this problem instead of the more natural "Tribonacci" recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$?*

Because the denominator $1 - x - x^2 - x^3$ doesn't factor nicely; it has three messy irrational roots.