Math 491, Problem Set #5: Solutions

Let a_n be the number of domino tilings of a 3-by-2n rectangle, and let b_n be the number of domino tilings of a 3-by-$(2n+1)$ rectangle from which a corner square has been removed. We showed in class that $a_n = a_{n-1} + 2b_{n-1}$ and $b_n = a_{n-1} + 3b_{n-1}$ for all $n \geq 1$.

1. *Introduce*

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

and

$$B(x) = b_0 + b_1 x + b_x x^2 + \ldots.$$

Write down two algebraic relations between $A(x)$ and $B(x)$ that represent the two recurrence relations (taking care to incorporate the boundary conditions correctly), and solve for $A(x)$ and $B(x)$.

The coefficient of x^n in $A(x) - xA(x) - 2xB(x)$ is $a_n - a_{n-1} - 2b_{n-1} = 0$ when $n \geq 1$ and is $a_0 = 1$ when $n = 0$; the coefficient of x^n in $B(x) - xA(x) - 3xB(x)$ is $b_n - a_{n-1} - 3b_{n-1} = 0$ when $n \geq 1$ and is $b_0 = 1$ when $n = 0$. So we have $(1-x)A(x) - 2xB(x) = 1$ and $xA(x) + (3x-1)B(x) = -1$. Solving, we get

$$A(x) = \frac{1-x}{1-4x+x^2}$$

and

$$B(x) = \frac{1}{1-4x+x^2}.$$

The roots of the denominator of $A(x)$ are $2 \pm \sqrt{3}$, whose reciprocals are one another; so the coefficients of $A(x)$ can be expressed in the form $a_n = C\alpha^n + D\beta^n$ where $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$ and where C, D are some undetermined constants (calculated in problem 2).

2. *We also saw in class that*

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Use linear algebra to derive a formula for a_n.

The eigenvalues of the matrix are the roots of $(\lambda-1)(\lambda-3)-(-2)(-1) = 0$, or $\lambda^2 - 4\lambda + 1$. Hence for any row vector v and column vector w (both of length 2), the scalar $v\ M^n\ w$ must be of the form $C\alpha^n + D\beta^n$, where α, β are the roots of $\lambda^2 - 4\lambda + 1 = 0$ (say $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$) and the coefficients C, D are determined by the choice of v and w. Setting $v = [1, 0]$ and $w = [1, 1]^T$, we see that a_n must be given by such a formula. To solve for C and D, set $n = 0$ and $n = 1$ to get $C = \frac{1}{2} + \frac{\sqrt{3}}{6}$ and $D = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Hence

$$a_n = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)(2 + \sqrt{3})^n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)(2 - \sqrt{3})^n.$$