Math 491, Problem Set #6: Solutions

1. *There is a unique polynomial of degree d such that $f(k) = 2^k$ for $k = 0, 1, ..., d$. What is $f(d+1)$? What is $f(-1)$?*

   Suppose $g(k)$ is a polynomial of degree $m \geq 1$, so that its sequence of mth differences is constant. If we define $G(k) = g(k) + g(k-1) + \ldots + g(1)$ for all $k \geq 1$, then the first differences of G are the "zeroeth" differences of g, the second differences of G are the first differences of g, and so on, so that the sequence of $m+1$st difference of G is constant, implying that $G(k)$ is given by a polynomial of degree $m+1$ in k. This last assertion is true for $g(k-1)+g(k-2)+\ldots+g(0)+1$ as well, since it differs from $G(k)$ by the substitution of $k-1$ for k and the addition of the constant 1.

   In particular, we see that if f is a polynomial of degree $d-1$ with $f(k) = 2^k$ for $0 \leq k \leq d-1$, then the sum $F(k) = f(k-1) + f(k-2) + \ldots + f(0) + 1$ defines a polynomial function of degree d, and it is easy to see that if f satisfies the property that characterizes f_{d-1}, F satisfies the property that characterizes f_d. Hence we have

   $$f_d(k) = f_{d-1}(k-1) + f_{d-1}(k-2) + \ldots + f_{d-1}(0) + 1$$

   for all $k \geq 0$ (not just $0 \leq k \leq d$), with the proviso that in the case $k = 0$, the only term on the right hand side is the 1.

   Putting $k = d+1$, we have $f_d(d+1) = f_{d-1}(d) + f_{d-1}(d-1) + \ldots + f_{d-1}(0) + 1 = f_{d-1}(d) + 2^{d-1} + \ldots + 1 + 1 = f^{d-1}(d) + 2^d$. That is, the sequence $f_0(1), f_1(2), f_2(3), \ldots,$ has the sequence $1, 2, 4, \ldots$ as its sequence of first differences, from which it follows (say by induction) that $f_{d-1}(d) = 2^d - 1$.

   On the other hand, for each fixed d the relation $f_d(k) - f_d(k-1) = f_{d-1}(k-1)$ holds for all k, since it holds for all positive k and since both sides of the equation are polynomials. Hence we have $f_d(0) - f_d(-1) = f_{d-1}(-1)$. Rewriting this as $f_d(-1) = f_d(0) - f_{d-1}(-1)$ and using the fact that $f_d(0) = 1$, we have $f_d(-1) = 1 - f_{d-1}(-1)$, from which it follows (say by induction) that $f_d(-1) = 1$ when d is even and 0 when d is odd. (Or, if you prefer formulas, $f_d = (1 + (-1)^n)/2$.

Note that you don't need to have an explicit formula for $f_d(k)$ in order to solve this problem!