On matchings of the 2-by-2-by-n cube snake

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1 Introduction

In considering perfect matchings of a 2-*by*-2-*by*-n cube snake, we came upon an interesting fact that each number of the sequence was either a perfect square or twice a perfect square. Namely, if n is an odd number, then the number of perfect matchings is twice a square, and if n is an even number, then the number of perfect matchings is a perfect square. The number of perfect matchings for a 2-*by*-2-*by*-n cube snake is given by the sequence $\{a\} = \{1, 2, 9, 32, 121, 450, 1681, 6272, \ldots\}$ $for\ n \geq 0$. This sequence, and some interesting properties of it, can be found on Sloane's Online Encyclopedia of Integer Sequences(A006253). The generating function for the sequence is given by

$$A(w) = \frac{2 + 3w + w^2}{1 - 3w - 3w^2 + w^3} = \sum_{i=0}^{\infty} k_i w^i \tag{1}$$

where k_i is the number of perfect matchings of the 2-*by*-2-*by*-i cube snake.

2 Bijection between perfect matchings of different combinatorial objects

Define the sequence $\{e\}$ and the sequence $\{o\}$ to be the sequences given by the even and odd terms of the sequence $\{a\}$, respectively. Then $\{e\} = \{1, 9, 121, 1681, 23409, 326041, \ldots\}$ and $\{o\} = \{2, 32, 450, 6272, 87362, 1216800, \ldots\}$. Then since $\{e\}$ has the property that each term e_i is a perfect square, we can take the square root of each term to get the first few terms of a new sequence $\{\sqrt{e}\} = \{1, 3, 11, 41, 153, 571, \ldots\}$. Analogously, since $\{o\}$ has the property that each term o_i is twice a perfect square, we can define a new sequence $\{\sqrt{\frac{o}{2}}\} = \{1, 4, 15, 56, 209, 780, \ldots\}$ where each term $\sqrt{\frac{o}{2}}_i$ is the term $\sqrt{\frac{o_i}{2}}$. What is interesting here is that the first terms of each of these sequences seem to define sequences of the matchings of different combinatorial objects. Namely, $\sqrt{e}_i$ coincides with the number of perfect matchings of a 3-*by*-$2i$ grid and $\sqrt{\frac{o}{2}}_i$ coincides with the number of perfect matchings of a 3-*by*-$(2i + 1)$ grid with a corner vertex removed (it also coincides with the number of spanning trees of the 2-*by*-n grid). These observances naturally led to the following conjecture.

Conjecture 1 *There exists a bijective map that maps each perfect matching of a* 2-*by*-2-*by*-$2n$ *cube snake to an ordered pair of perfect matchings of the* 3-*by*-$2n$ *grid. Additionally, there exists a bijective map that maps half of the perfect matchings of a* 2-*by*-2-*by*-$(2n + 1)$ *cube snake to an ordered pair of perfect matchings of the* 3-*by*-$(2n + 1)$ *grid for* $n \geq 0$

3 Introducing weighted edges to the cube snake

In attempting to prove conjecture (1), it was natural to try to extract more information from the combinatorics of perfect matchings of the cube snake.

Conjecture 2 *Consider the cube snake given weights of x, y and z to edges of the cube snake that exist in the conventional axes x, y and z of R^3, respectively. Then the generating function of perfect matchings of the 2-by-2-by-n cube snake involving the formal variables x, y and z is*

$$A(w,x,y,z) = \frac{y^2 + z^2 + x^2(x^2+y^2+z^2)w - x^6w^2}{1-(x^2+y^2+z^2)w - x^2(x^2+y^2+z^2)w^2 + x^6w^3} \tag{2}$$

Notice that the first few coefficients of the terms of the Taylor expansion of $A(w, x, y, z)$ are perfect squares in the polynomials of x, y and z for the even terms and the product of $(y^2 + z^2)$ and a perfect square of a polynomial in x, y and z for odd terms. Notice that setting $x = y = z = 1$ gives the generating function (1). If conjecture (2) is true, then assigning the correct corresponding weights to the edges of the 3-*by*-$2i$ and 3-*by*-$(2i+1)$ grids should allow us to gain headway in proving conjecture (1).

4 On proving conjecture (2)

Proving conjecture (2) should be analogous to the methods used to prove that equation 1 is the generating function for the cube snake. This was proven (at least by myself) by setting up two recurrence relations $\{a\}$ and $\{b\}$ where $\{a\}$ defines the recurrence for the 2-*by*-2-*by*-n cube snake and $\{b\}$ defines the recurrence of the 2-*by*-2-*by*-n snake with two adjacent vertices on the outermost box removed. One can find easily the recurrence $\{a\}$ in terms of combinations of terms of the sequences $\{a\}$ and $\{b\}$. One can also easily find the recurrence $\{b\}$ in terms of combinations of terms of the sequences $\{a\}$ and $\{b\}$. From there, we just solve the two equations for the two unknowns and come up with a recurrence for $\{a\}$ that depends only on other terms of the sequence $\{a\}$. It should not take more than considering the formal variables x, y and z in the original setup of the recurrences to prove conjecture (2).