ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2019, Vol. 305, pp. 305–317. © Pleiades Publishing, Ltd., 2019.
Russian Text © The Author(s), 2019, published in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2019, Vol. 305, pp. 330–343.

An Elementary Approach to the Study of Somos Sequences

A. V. Ustinov [a]

Received August 9, 2018; revised December 7, 2018; accepted December 10, 2018

Abstract—An elementary approach (not involving the theory of elliptic functions) is proposed to the proof of the main properties of the Somos-4 sequence.

DOI: 10.1134/S0081543819030180

1. INTRODUCTION

The Somos-k sequence is a number sequence $\{s_n\}$ satisfying the kth-order ($k \ge 2$) quadratic recurrence relation

$$s_{n+k}s_n = \sum_{1\le j\le k/2} \alpha_j s_{n+k-j}s_{n+j}, \tag{1.1}$$

where α_j ($1 \le j \le k/2$) are constants. Unless specified otherwise, we will assume that the elements of the sequence $\{s_n\}$ do not vanish and are therefore defined for all integers n.

Among the Somos sequences, there is an important class of Laurent phenomenon sequences: all their terms are *Laurent polynomials* in the initial conditions, i.e., $s_n \in \mathbb{Z}[s_1^{\pm1},\dots,s_k^{\pm1},\alpha_1,\dots,\alpha_{\lfloor k/2\rfloor}]$. This class is known to include the Somos-k sequences with $k = 4,5,6,7$ and some higher order sequences. In particular, this implies that for $s_1 = \ldots = s_k = \alpha_1 = \ldots = \alpha_{\lfloor k/2\rfloor} = 1$ the Somos-k sequences ($k = 4,5,6,7$) are integer. For $k = 4,5$, this fact admits a simple proof (see [7]), which, unfortunately, does not clarify the nature of these sequences at all. The fact that the Somos-6 and Somos-7 sequences are integer was first given a computer-based proof. When describing the history of the Somos sequences, Gale [7] compared these results with the proof of the four color theorem.

While developing the theory of cluster algebras, Fomin and Zelevinsky [6] proposed a general approach to proving the Laurent phenomenon for the Somos sequences. Subsequently (see [1]), the Laurent phenomenon was systematically studied for the sequences defined by recurrence relations of the form

$$x_{n+k}x_n = P(x_{n+1},\dots,x_{n+k-1}), \tag{1.2}$$

where P is a polynomial of arbitrary degree. Sufficient conditions for the Somos sequences to be integer are the subject of separate investigation (see [11]).

Among the Somos sequences, one can also distinguish a class consisting of *finite rank* sequences. A sequence $\{s_n\}_{n=-\infty}^{\infty}$ is said to be of (finite) rank r if $r = \max\{r_0, r_1\}$ with r_0 and r_1 being the ranks of the infinite matrices $(s_{m+n}s_{m-n})_{m,n=-\infty}^{\infty}$ and $(s_{m+n+1}s_{m-n})_{m,n=-\infty}^{\infty}$, respectively. Equivalently, r_0 and r_1 can be defined as the least positive integers for which there exist representations

$$s_{m+n}s_{m-n} = \sum_{j=1}^{r_0} f_j(m)g_j(n), \qquad s_{m+n+1}s_{m-n} = \sum_{j=1}^{r_1} \widetilde{f}_j(m)\widetilde{g}_j(n).$$

[a] Khabarovsk Division of the Institute of Applied Mathematics, Far Eastern Branch of the Russian Academy of Sciences, ul. Dzerzhinskogo 54, Khabarovsk, 680000 Russia.

E-mail addresses: ustinov@iam.khv.ru, ustinov.alexey@gmail.com

In [2], Avdeeva and Bykovskii proved that the rank of the product of two sequences does not exceed the product of the ranks of these sequences. Therefore, the sequences of finite rank form a ring with the simplest elements given by the (rank 2) sequence $s_n = n$ and the (rank 1) sequence

$$s_n = e^{an^2+bn+c}. \tag{1.3}$$

In [2], Avdeeva and Bykovskii also proved that finite rank sequences cannot grow too fast; namely, the estimate $|s_n| \le c_1 e^{c_2 n^2}$ always holds for some positive c_1 and c_2. Some sequences defined by relation (1.2) exhibit the Laurent phenomenon but have higher growth order. Therefore, the class of Laurent phenomenon sequences is wider than that of finite rank sequences. However, in the case of Somos sequences, it is natural to expect that the class of finite rank sequences is close to the class of Laurent phenomenon sequences.

In a more general situation (see [2–4]), one considers hyperelliptic systems of sequences consisting of pairs $\{a_n\}_{n=-\infty}^{\infty}, \{b_n\}_{n=-\infty}^{\infty}$ that can be represented as

$$a_{m+n}b_{m-n} = \sum_{j=1}^{r_0} f_j(m)g_j(n), \qquad a_{m+n+1}b_{m-n} = \sum_{j=1}^{r_1} \widetilde{f}_j(m)\widetilde{g}_j(n).$$

In the continuous case, instead of sequences one studies hyperelliptic systems of functions (see [18, 3, 12–14]) consisting of pairs of functions f, g satisfying the functional equation

$$f(x+y)g(x-y) = \sum_{j=1}^{r} \varphi_j(x)\psi_j(y).$$

The terminology goes back to the theory of integral equations, in which finite rank kernels are used to approximate arbitrary kernels of integral operators (see [17, Sect. 69] and [21, § 3.6]).

The first nontrivial example is given by the Somos-4 sequence, for which the recurrence relation can be written as

$$s_{n+2}s_{n-2} = \alpha s_{n+1}s_{n-1} + \beta s_n^2. \tag{1.4}$$

An arbitrary sequence of rank 1 has the form (1.3). It satisfies the recurrence relation (1.4) if $e^{8a} = \alpha e^{2a} + \beta$. In other cases, the Somos-4 sequence has rank 2. This follows, for example, from the general formula

$$s_n = AB^n \frac{\sigma(z_0 + nz)}{\sigma(z)^{n^2}}, \tag{1.5}$$

which was found independently by Hone [8] and Swart [20]; here $\sigma(z) = \sigma(z; g_2, g_3)$ is the Weierstrass sigma function associated with the curve

$$y^2 = 4x^3 - g_2x - g_3 \tag{1.6}$$

(see [23]). The sequence of all integers $s_n = n$ ($\alpha = 4$ and $\beta = -3$) and the sequence of Fibonacci numbers $s_n = F_n$ ($\alpha = -1$ and $\beta = 2$) are the simplest particular cases of the Somos-4 sequences.

The Somos-6 sequence in general has rank 4, which also follows from the general formulas (see [10, 5]).

The *elliptic divisibility sequences* $\{w_n\}_{n=-\infty}^{\infty}$, which are integer sequences defined by the conditions

$$w_{-n} = -w_n \qquad (w_0 = 0), \qquad w_1 = 1, \qquad w_2, w_3, w_4 \in \mathbb{Z}, \qquad w_2 \mid w_4, \tag{1.7}$$

$$w_1^2 w_{n+2}w_{n-2} = w_{n+1}w_{n-1}w_2^2 - w_1w_3w_n^2 \tag{1.8}$$

(see [22]), are a particular case of the Somos-4 sequences.

If the sequence $\{w_n\}$ is not required to be integer, then the general solution can be expressed as

$$w_n = \frac{\sigma(nz)}{\sigma(z)^{n^2}}. \tag{1.9}$$

In [15], Ma proved that the theorem on the rank of the Somos-4 sequence follows from the addition theorems

$$w_1^2 s_{n+k}s_{n-k} = \begin{vmatrix} s_{n+1}w_k & s_n w_{k+1} \\ s_n w_{k-1} & s_{n-1}w_k \end{vmatrix}, \qquad w_1 w_2 s_{n+k-1}s_{n-k} = \begin{vmatrix} s_{n+2}w_{k+1} & s_{n+1}w_{k+2} \\ s_n w_{k-1} & s_{n-1}w_k \end{vmatrix}. \tag{1.10}$$

Formulas (1.10) were obtained by van der Poorten and Swart in [16]. They follow from (1.5) and (1.9), but they can also be derived without invoking elliptic functions. The paper [16] contains an elementary proof of formulas (1.10) that is based on nontrivial symmetry considerations.

In the present paper, we propose an elementary approach (without invoking Weierstrass functions) to studying the properties of the Somos-4 sequences. We give a direct inductive proof of the theorem on the rank of the Somos-4 sequence (see Theorem 3.7). Using this theorem, we prove various addition formulas similar to the well-known identities for the Weierstrass functions (see Theorems 4.1 and 4.6). Note also that an elliptic divisibility sequence $\{w_n\}$ arises naturally in the proof of the theorem on the rank of the Somos-4 sequence.

2. FIRST INTEGRALS

The Somos sequences have an obvious symmetry: if a sequence $\{s_n\}$ satisfies relation (1.1), then the sequence

$$\widetilde{s}_n = AB^n s_n \qquad (A, B \neq 0) \tag{2.1}$$

satisfies the same relation. Therefore, to simplify further arguments, it is natural to introduce the new variables

$$f_n = \frac{s_{n-1}s_{n+1}}{s_n^2},$$

which are invariant under the action of the gauge group (2.1). In terms of the new variables, the recurrence relation (1.4) reads

$$f_{n-1}f_n^2 f_{n+1} = \alpha f_n + \beta. \tag{2.2}$$

Given two sequences $\{a_n\}_{n=-\infty}^{\infty}$ and $\{b_n\}_{n=-\infty}^{\infty}$, define the matrices $M_{a,b}^{(0)} = (a_{m+n}b_{m-n})_{m,n=-\infty}^{\infty}$ and $M_{a,b}^{(1)} = (a_{m+n+1}b_{m-n})_{m,n=-\infty}^{\infty}$. Denote by

$$M_{a,b}^{(0)}\begin{pmatrix} m_1, m_2, \dots, m_k \\ n_1, n_2, \dots, n_k \end{pmatrix} \qquad \text{and} \qquad M_{a,b}^{(1)}\begin{pmatrix} m_1, m_2, \dots, m_k \\ n_1, n_2, \dots, n_k \end{pmatrix}$$

the finite submatrices of $M_{a,b}^{(0)}$ and $M_{a,b}^{(1)}$, respectively, that are composed of the elements in columns with numbers $m_1, \dots, m_k$ and rows with numbers $n_1, \dots, n_k$. If we allow the parameters $m_1, \dots, m_k$ and $n_1, \dots, n_k$ to take half-integer values simultaneously, then the matrices will be related by the equality

$$M_{a,b}^{(0)}\begin{pmatrix} m_1+\frac{1}{2}, m_2+\frac{1}{2}, \dots, m_k+\frac{1}{2} \\ n_1+\frac{1}{2}, n_2+\frac{1}{2}, \dots, n_k+\frac{1}{2} \end{pmatrix} = M_{a,b}^{(1)}\begin{pmatrix} m_1, m_2, \dots, m_k \\ n_1, n_2, \dots, n_k \end{pmatrix}.$$

Set

$$D_{a,b}^{(0,1)}\begin{pmatrix} m_1, m_2, \dots, m_k \\ n_1, n_2, \dots, n_k \end{pmatrix} = \det M_{a,b}^{(0,1)}\begin{pmatrix} m_1, m_2, \dots, m_k \\ n_1, n_2, \dots, n_k \end{pmatrix}.$$

A key to the proof of the properties of the Somos-4 sequence is that this sequence has a first integral, i.e., a shift-invariant quantity

$$T_n = f_n f_{n+1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n+1}}\right) + \frac{\beta}{f_n f_{n+1}}, \tag{2.3}$$

which was found in [20, pp. 161–162] (see also [8, 9]). The explicit form of this first integral can be obtained as follows. Assuming that Theorem 3.7 (see below) holds, we know that the rank of each of the matrices $M^{(0)}_{s,s}$ and $M^{(1)}_{s,s}$ is 2. Therefore, the equalities

$$D^{(0)}_{s,s}\binom{m_1, m_2, m_3}{n_1, n_2, n_3} = D^{(1)}_{s,s}\binom{m_1, m_2, m_3}{n_1, n_2, n_3} = 0 \tag{2.4}$$

must hold for any integers m_1, m_2, m_3 and n_1, n_2, n_3. The simplest of equalities (2.4) that needs to be substantiated is the equality $D^{(1)}_{s,s}\binom{n,3,2}{2,1,0} = 0$. By equivalence transformations, it can be reduced to

$$\beta s_n s_{n+1}\begin{vmatrix} s_5 s_2 & s_4 s_3 \\ s_4 s_1 & s_3 s_2 \end{vmatrix}(T_n - T_1) = 0.$$

Therefore, if formulas (2.4) are indeed valid, then T_n must be independent of n. Having found the explicit form of the first integral, one can prove its invariance using the basic recurrence relation (1.4).

Lemma 2.1. *The sequence $\{f_n\}$ defined by the recurrence relation* (2.2) *has the first integral*

$$T = f_n f_{n+1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n+1}}\right) + \frac{\beta}{f_n f_{n+1}}. \tag{2.5}$$

Moreover, for any integer n,

$$T = f_n(f_{n-1} + f_{n+1}) + \frac{\alpha}{f_n}. \tag{2.6}$$

For completeness, we present a proof of this assertion (see [20, 8]).

Proof. Dividing the equalities

$$f_{n-1} f_n^2 f_{n+1} - \alpha f_n = \beta = f_n f_{n+1}^2 f_{n+2} - \alpha f_{n+1}$$

by $f_n f_{n+1}$, we find

$$f_{n-1} f_n - \frac{\alpha}{f_{n+1}} = \frac{\beta}{f_n f_{n+1}} = f_{n+1} f_{n+2} - \frac{\alpha}{f_n}.$$

Adding the terms $f_n f_{n+1} + \alpha(1/f_n + 1/f_{n+1})$ to each of the expressions obtained, we arrive at the relations

$$f_n f_{n-1} + \frac{\alpha}{f_n} + f_n f_{n+1} = T_n = f_{n+1} f_n + \frac{\alpha}{f_{n+1}} + f_{n+1} f_{n+2}, \tag{2.7}$$

where T_n is defined by (2.3). Relations (2.7) imply that T_n is independent of the number n and that representation (2.6) is valid. □

Remark 2.2. For $f_{n+1} = x$ and $f_n = y$, equality (2.5) reduces to the equation of an elliptic curve,

$$x^2 y^2 - Txy + \alpha(x + y) + \beta = 0; \tag{2.8}$$

this explains the need to use elliptic functions in the formula for the general solution (1.5). The pair $(x, y) = (f_{n-1}, f_n)$ also satisfies equation (2.8); hence, the numbers $x_{1,2} = f_{n\pm 1}$ are the roots of the quadratic equation

$$x^2 f_n^2 - x(Tf_n - \alpha) + (\alpha f_n + \beta) = 0. \tag{2.9}$$

Thus, equalities (2.5) and (2.6) can be interpreted as Vieta's theorem for the quadratic equation (2.9).

Below, in addition to the variables f_n, we will need other gauge-invariant variables

$$h_n = \frac{s_{n+2}s_{n-1}}{s_{n+1}s_n} = f_{n+1}f_n. \tag{2.10}$$

Using these variables, we can rewrite the recurrence relation (2.2) as $h_{n-1}h_n = \alpha f_n + \beta$, which allows us to express f_n in terms of the elements of the sequence $\{h_n\}$:

$$f_n = \frac{h_{n-1}h_n - \beta}{\alpha}. \tag{2.11}$$

Lemma 2.3. *The elements of the sequence* $\{h_n\}$ *satisfy the relations*

$$h_{n-1}h_nh_{n+1} = -\beta h_n + \alpha^2 + T\beta, \tag{2.12}$$

$$h_n(h_{n-1} + h_{n+1}) = -h_n^2 + Th_n + \beta. \tag{2.13}$$

Proof. Using equalities (2.10) and (2.11), we rewrite formula (2.6) as

$$T = h_{n-1} + h_n + \frac{\alpha^2}{h_nh_{n-1} - \beta}. \tag{2.14}$$

This equality is equivalent to the fact that the quadratic equation

$$z^2h_n + z(h_n^2 - \beta - Th_n) + (\alpha^2 + T\beta - \beta h_n) = 0 \tag{2.15}$$

has a root $z = h_{n-1}$. Replacing n by $n+1$ in (2.14), we find that the second root of equation (2.15) is $z = h_{n+1}$. Equalities (2.12) and (2.13) are a consequence of Vieta's theorem applied to equation (2.15). ☐

The recurrence relation (2.12) differs little in form from equation (2.2). Below we will need the following analog of Lemma 2.1 for sequences defined by the formula

$$h_{n-1}h_nh_{n+1} = \widetilde{\alpha}h_n + \widetilde{\beta}. \tag{2.16}$$

Lemma 2.4. *The sequence* $\{h_n\}$ *defined by the recurrence relation* (2.16) *has the first integral*

$$\widetilde{T} = h_n + h_{n-1} + \widetilde{\alpha}\left(\frac{1}{h_n} + \frac{1}{h_{n-1}}\right) + \frac{\widetilde{\beta}}{h_nh_{n-1}}. \tag{2.17}$$

Moreover, for any integer n,

$$\widetilde{T} = h_{n-1} + h_n + h_{n+1} + \frac{\widetilde{\alpha}}{h_n}. \tag{2.18}$$

Proof. Dividing the equalities

$$h_{n-2}h_{n-1}h_n - \widetilde{\alpha}h_{n-1} = \widetilde{\beta} = h_{n-1}h_nh_{n+1} - \widetilde{\alpha}h_n$$

by $h_{n-1}h_n$, we obtain

$$h_{n-2} - \frac{\widetilde{\alpha}}{h_n} = \frac{\widetilde{\beta}}{h_{n-1}h_n} = h_{n+1} - \frac{\widetilde{\alpha}}{h_{n-1}}.$$

Adding $h_{n-1}+h_n+\widetilde{\alpha}(1/h_{n-1}+1/h_n)$ to each of the expressions obtained, we arrive at the relations

$$h_{n-2}+h_{n-1}+h_n+\frac{\widetilde{\alpha}}{h_{n-1}}=\widetilde{T}_n=h_{n-1}+h_n+h_{n+1}+\frac{\widetilde{\alpha}}{h_n}, \tag{2.19}$$

where $\widetilde{T}_n$ denotes the right-hand side of (2.17). Relations (2.19) imply that $\widetilde{T}_n$ are independent of the number n and that representation (2.18) is valid. $\square$

Remark 2.5. Lemma 2.4 implies, in particular, that equality (2.13) is a consequence of relation (2.12).

3. THEOREM ON THE RANK OF A MATRIX

Proposition 3.1. *Suppose that α, β, and T are algebraically independent over the field of rational numbers. Then the rank of the matrix $M^{(0)}_{s,s}$ is at most 2.*

Proof. Let us show that the kth column of the matrix $M^{(0)}_{s,s}$, where k is an arbitrary integer, can be represented as a linear combination of the zeroth and first columns. To this end, we construct two sequences $\{c_k\}_{k=1}^{\infty}$ and $\{d_k\}_{k=1}^{\infty}$ such that

$$s_{n+k}s_{n-k}=c_k s_{n+1}s_{n-1}-d_k s_n^2. \tag{3.1}$$

Introduce the new variables

$$F_k^{(n)}=\frac{s_{n+k}s_{n-k}}{s_n^2}. \tag{3.2}$$

In particular, $F_0^{(n)}=1$, $F_1^{(n)}=f_n$, and $F_2^{(n)}=\alpha f_n+\beta$. Then equality (3.1) can be rewritten as

$$F_k^{(n)}=c_k f_n-d_k. \tag{3.3}$$

Since $F_k^{(n)}=F_{-k}^{(n)}$, it suffices to restrict the analysis to nonnegative values of k. It is obvious that equality (3.3) will hold for $k=1$ and $k=2$ if we set

$$c_1=1,\qquad d_1=0,\qquad c_2=\alpha,\qquad d_2=-\beta.$$

Given the values of c_{k-1}, d_{k-1}, c_k, and d_k, one can find the coefficients c_{k+1} and d_{k+1} using the equality

$$F_{k+1}^{(n)}=\frac{F_k^{(n-1)}F_k^{(n+1)}}{F_{k-1}^{(n)}}f_n^2,$$

which follows from the definition (3.2).

Let us transform this equality with the help of (3.3):

$$c_{k+1}f_n-d_{k+1}=\frac{(c_k f_{n-1}-d_k)(c_k f_{n+1}-d_k)}{c_{k-1}f_n-d_{k-1}}f_n^2=\frac{c_k^2 f_{n-1}f_n^2 f_{n+1}-c_k d_k f_n^2(f_{n-1}+f_{n+1})+d_k^2 f_n^2}{c_{k-1}f_n-d_{k-1}}.$$

Applying relations (2.2) and (2.6) to the right-hand side,

$$c_{k+1}f_n-d_{k+1}=\frac{c_k^2(\alpha f_n+\beta)-c_k d_k(Tf_n-\alpha)+d_k^2 f_n^2}{c_{k-1}f_n-d_{k-1}},$$

we see that the equality $F_{k+1}^{(n)}=c_{k+1}f_n-d_{k+1}$ is equivalent to

$$d_k^2 f_n^2+c_k f_n(\alpha c_k-Td_k)+c_k(\beta c_k+\alpha d_k)-(c_{k+1}f_n-d_{k+1})(c_{k-1}f_n-d_{k-1})=0.$$

Equating the coefficients of all powers of f_n to zero, we obtain an overdetermined system of equations with unknowns c_{k+1} and d_{k+1}:

$$c_{k+1}c_{k-1} = d_k^2, \tag{3.4}$$

$$c_{k-1}d_{k+1} + c_{k+1}d_{k-1} = -c_k(\alpha c_k - Td_k), \tag{3.5}$$

$$d_{k-1}d_{k+1} = c_k(\beta c_k + \alpha d_k). \tag{3.6}$$

For $k = 2$, the last equation degenerates and the values $c_3 = \beta^2$ and $d_3 = -\alpha(\alpha^2 + T\beta)$ are determined from the first two equations. For $k \geq 3$, we need to show that system (3.4)–(3.6) is consistent. We will assume that the values of c_{k+1} and d_{k+1} are determined from equations (3.4) and (3.6), i.e.,

$$c_{k+1} = \frac{d_k^2}{c_{k-1}}, \qquad d_{k+1} = \frac{c_k(\beta c_k + \alpha d_k)}{d_{k-1}}. \tag{3.7}$$

Substituting them into equation (3.5), we obtain the consistency condition for the system,

$$g_{k-1}g_k + \alpha\left(\frac{1}{g_{k-1}} + \frac{1}{g_k}\right) + \frac{\beta}{g_{k-1}g_k} = T, \tag{3.8}$$

where $g_k = d_k/c_k$. The equalities $g_2 = -\beta/\alpha$ and $g_3 = -\alpha(\alpha^2 + T\beta)/\beta^2$ imply that (3.8) holds for $k = 3$. Dividing the second equality in (3.7) by the first, we find that the sequence $\{g_k\}$ satisfies a recurrence relation that coincides with the recurrence relation for the elements of the sequence $\{f_k\}$:

$$g_{k-1}g_k^2 g_{k+1} = \alpha g_k + \beta.$$

Therefore, the validity of (3.8) follows from Lemma 2.1.

To complete the proof of Proposition 3.1, we should show that the sequences $\{c_k\}$ and $\{d_k\}$ are well defined by the recurrence relations (3.7). As pointed out above, the sequence $s_n = n$ is a Somos-4 sequence and satisfies the recurrence equation $s_{n+2}s_{n-2} = 4s_{n+1}s_{n-1} - 3s_n^2$. It satisfies relation (3.3) for $c_k = k^2$ and $d_k = k^2 - 1$. Therefore, the elements of the sequences $\{c_k\}$ and $\{d_k\}$ calculated by formulas (3.7) are rational functions of α, β, and T that are different from the identical zero. Thus, the algebraic independence of α, β, and T guarantees that the elements of the sequences $\{c_k\}$ and $\{d_k\}$ (except for $d_1 = 0$) do not vanish and hence can indeed be calculated by formulas (3.7). □

The sequence $\{f_n\}$ was initially constructed using the sequence $\{s_n\}$. Therefore, an analogous sequence $\{g_n\}$ should also correspond to some Somos-4 sequence, which we will denote by $\{w_n\}$ (it will be clear from further considerations that this is the elliptic divisibility sequence mentioned above). The equality $g_k = w_{k-1}w_{k+1}/w_k^2$ implies that for fixed values of $g_k = d_k/c_k$, the elements of the sequence $\{w_n\}$ are defined uniquely up to multiplication by a geometric progression. The elements of the sequence $\{w_n\}$ take the simplest form if we assume that $w_k^2 = c_k$ and $w_{k-1}w_{k+1} = d_k$ $(k = 1, 2, \dots)$. Then, based on the initial conditions $w_0w_2 = 0$, $w_1^2 = 1$, and $w_2^2 = \alpha$, it is natural to choose the first elements of the sequence as follows:

$$w_0 = 0, \qquad w_1 = 1, \qquad w_2 = -\sqrt{\alpha}. \tag{3.9}$$

The other elements are uniquely determined from the equality $w_{k-1}w_{k+1} = d_k$ $(k = 2, 3, \dots)$. For example,

$$w_3 = -\beta, \qquad w_4 = \sqrt{\alpha}(\alpha^2 + T\beta), \qquad w_5 = \beta^3 - \alpha^2(\alpha^2 + T\beta). \tag{3.10}$$

Remark 3.2. In terms of the sequence $\{w_n\}$, equality (3.3) can be written as

$$w_1^2 s_{n+k}s_{n-k} = w_k^2 s_{n+1}s_{n-1} - w_{k+1}w_{k-1}s_n^2, \tag{3.11}$$

which coincides with the first formula in (1.10). Formula (3.11) can also be applied, in particular, to the sequence $s_n = w_n$. Therefore,

$$w_1^2 w_{n+k} w_{n-k} = w_k^2 w_{n+1} w_{n-1} - w_{k+1} w_{k-1} w_n^2. \tag{3.12}$$

Substituting first $k = 0$ and $n = 1$ into this formula, we find that $w_{-1} = -1$. Setting then $n = 0$, we find that for negative numbers the sequence $\{w_n\}$ should be defined in an odd manner: $w_{-n} = -w_n$.

Remark 3.3. The factor $w_1^2 = 1$ in the recurrence relation (1.8) is added to ensure that the sum of squared indices in each term is the same. This condition also holds for other "addition theorems" for Somos sequences (see formulas (1.10), (4.1), and (4.4)–(4.6) below). This is associated with the origin of the Somos sequences (see Somos's notes [19]).

Proposition 3.4. *Let numbers α, β, and T be algebraically independent over the field of rational numbers. Then the rank of the matrix $M_{s,s}^{(1)}$ is at most* 2.

Proof. Just as in the proof of Proposition 3.1, we show that the kth column of $M_{s,s}^{(1)}$, where k is an arbitrary integer, can be represented as a linear combination of the zeroth and first columns. To this end, we produce two sequences $\{u_k\}_{k=-\infty}^{\infty}$ and $\{v_k\}_{k=-\infty}^{\infty}$ such that

$$s_{n+k+1} s_{n-k} = u_k s_{n+2} s_{n-1} - v_k s_n s_{n+1}. \tag{3.13}$$

Obviously, it suffices to take

$$u_0 = 0, \qquad v_0 = -1, \qquad u_1 = 1, \qquad v_1 = 0$$

as the initial conditions. Equality (2.12) means that $u_2 = -\beta$ and $v_2 = -(\alpha^2 + T\beta)$.

Introduce the new variables

$$G_k^{(n)} = \frac{s_{n+k+1} s_{n-k}}{s_n s_{n+1}}. \tag{3.14}$$

In particular, $G_0^{(n)} = 1$, $G_1^{(n)} = h_n$, and $G_2^{(n)} = h_{n-1} h_n h_{n+1} = -\beta h_n + \alpha^2 + T\beta$. Then equality (3.13) can be rewritten as

$$G_k^{(n)} = u_k h_n - v_k. \tag{3.15}$$

Since $G_{-k}^{(n)} = G_{k-1}^{(n)}$, to prove the lemma it suffices to define $\{u_k\}$ and $\{v_k\}$ for nonnegative values of k and set $u_{-k} = u_{k-1}$ and $v_{-k} = v_{k-1}$ $(k \ge 1)$.

Given the values of u_{k-1}, v_{k-1}, u_k, and v_k, we find the coefficients u_{k+1} and v_{k+1} from the equality

$$G_{k+1}^{(n)} = \frac{G_k^{(n-1)} G_k^{(n+1)}}{G_{k-1}^{(n)}} h_n,$$

which follows from the definition (3.14). Let us transform this equality with the help of (3.15):

$$(u_{k+1} h_n - v_{k+1})(u_{k-1} h_n - v_{k-1}) = (u_k h_{n-1} - v_k)(u_k h_{n+1} - v_k) h_n.$$

Using formulas (2.12) and (2.13), we can reduce this equality to

$$h_n^2 (u_{k+1} u_{k-1} - u_k v_k) - h_n \bigl(u_{k+1} v_{k-1} + u_{k-1} v_{k+1} - \beta u_k^2 - u_k v_k T + v_k^2\bigr) \\ + \bigl(v_{k-1} v_{k+1} - u_k^2 (\alpha^2 + T\beta) + u_k v_k \beta\bigr) = 0.$$

Equating the coefficients of all powers of h_n to zero, we obtain an overdetermined system with unknowns u_{k+1} and v_{k+1} $(k \ge 2)$:

$$u_{k+1} u_{k-1} = u_k v_k, \tag{3.16}$$

$$u_{k+1}v_{k-1}+u_{k-1}v_{k+1}=\beta u_k^2+u_kv_kT-v_k^2, \tag{3.17}$$

$$v_{k-1}v_{k+1}=u_k^2(\alpha^2+T\beta)-u_kv_k\beta. \tag{3.18}$$

Substituting the values of u_{k+1} and v_{k+1} expressed from (3.16) and (3.18) into equation (3.17), we obtain the consistency condition for the system,

$$\xi_{k-1}\xi_k+\alpha\biggl(\frac{1}{\xi_{k-1}}+\frac{1}{\xi_k}\biggr)+\frac{\beta}{\xi_{k-1}\xi_k}=T, \tag{3.19}$$

with $\xi_k=v_k/u_k$. In particular,

$$\xi_2=\frac{\alpha^2+T\beta}{\beta},\qquad \xi_3=\frac{\beta^3-\alpha^2(\alpha^2+T\beta)}{\beta(\alpha^2+T\beta)};$$

therefore, equality (3.19) holds for $k=3$. Dividing (3.18) by (3.16), we arrive at the recurrence relation for the elements of the sequence $\{\xi_k\}$:

$$\xi_{k-1}\xi_k^2\xi_{k+1}=-\beta\xi_k+\alpha^2+T\beta. \tag{3.20}$$

This relation coincides with (2.16) for $\widetilde{\alpha}=-\beta$ and $\widetilde{\beta}=\alpha^2+T\beta$. Thus, the validity of formula (3.19) (and, hence, the consistency of system (3.16)–(3.18)) follows from Lemma 2.4.

The fact that the sequences $\{u_k\}_{k=-\infty}^{\infty}$ and $\{v_k\}_{k=-\infty}^{\infty}$ are well defined by the recurrence relations (3.16) and (3.18) (i.e., no division by zero arises) is verified in the same way as in the proof of Proposition 3.1. □

Remark 3.5. The elements of the sequence $\{h_n\}$ are constructed on the basis of the Somos sequence $\{s_n\}$ by formula (2.10). Therefore, the sequence $\{\xi_n\}$ should also correspond to some Somos-4 sequence. It turns out that this is the same sequence $\{w_n\}$ that was described in Remark 3.2. For $k=2,3$, the equality $\xi_k=w_{k-1}w_{k+2}/w_kw_{k+1}$ is a consequence of (3.9) and (3.10), and for $k\ge 4$, it follows from the coincidence of the recurrence relations (2.16) and (3.20). Thus, equality (3.13) can be rewritten as

$$w_1w_2s_{n+k+1}s_{n-k}=w_kw_{k+1}s_{n+2}s_{n-1}-w_{k+2}w_{k-1}s_ns_{n+1}, \tag{3.21}$$

which coincides with the second formula in (1.10). In particular, for $s_n=w_n$,

$$w_1w_2w_{n+k+1}w_{n-k}=w_kw_{k+1}w_{n+2}w_{n-1}-w_{k+2}w_{k-1}w_nw_{n+1}. \tag{3.22}$$

Thus, the elements of the sequences $\{u_k\}$ and $\{v_k\}$ that appear in the proof of the proposition have the form

$$u_k=\frac{w_kw_{k+1}}{w_1w_2},\qquad v_k=\frac{w_{k+2}w_{k-1}}{w_1w_2}.$$

Remark 3.6. As shown in [11], formulas (3.12) and (3.22) imply that $w_{2n-1}\in\mathbb{Z}[\alpha^2,\beta,I]$ and $w_{2n}\in\sqrt{\alpha}\mathbb{Z}[\alpha^2,\beta,I]$, where $I=\alpha^2+\beta T$. Therefore, the sequence $\{w_n\}$ is uniquely defined by the triple of parameters (α,β,T). In particular, this implies that for the shifted sequence $\widetilde{s}_n=s_{n+t}$ the sequence $\{w_n\}$ will be the same.

Theorem 3.7. *The rank of an arbitrary Somos-*4 *sequence different from sequence* (1.3) *is* 2.

Proof. In the case where α, β, and T are algebraically independent over the field of rational numbers, the theorem follows from Propositions 3.1 and 3.4. In the general case, the algebraic relations (3.11) and (3.21), which express the linear dependence of the columns of the matrices $M_{s,s}^{(0)}$ and $M_{s,s}^{(1)}$, remain valid due to continuity arguments. □

4. ADDITION THEOREMS

Formulas (3.11) and (3.21) are particular cases of a more general addition theorem.

Theorem 4.1. *Let numbers k, l, m, and n be simultaneously integer or half-integer. Then*

$$s_{k+l}s_{m+n}w_{k-l}w_{m-n}+s_{k+n}s_{l+m}w_{k-n}w_{l-m}-s_{k+m}s_{l+n}w_{k-m}w_{l-n}=0. \tag{4.1}$$

In particular,

$$w_{k+l}w_{m+n}w_{k-l}w_{m-n}+w_{k+n}w_{l+m}w_{k-n}w_{l-m}-w_{k+m}w_{l+n}w_{k-m}w_{l-n}=0. \tag{4.2}$$

Proof. First, consider the case when $k+l+m+n\equiv 0 \pmod 2$. Passing to the parameters

$$a=\frac{k+l+m+n}{2},\qquad b=\frac{k+l-m-n}{2},\qquad c=\frac{k-l+m-n}{2},\qquad d=\frac{k-l-m+n}{2}, \tag{4.3}$$

we can rewrite equality (4.1) as

$$s_{a+b}s_{a-b}w_{c+d}w_{c-d}+s_{a+d}s_{a-d}w_{b+c}w_{b-c}-s_{a+c}s_{a-c}w_{b+d}w_{b-d}=0. \tag{4.4}$$

To check the validity of this equality, it suffices to apply formulas (3.11) and (3.12) to all products of the form $s_{x+y}s_{x-y}$ and $w_{x+y}w_{x-y}$, respectively.

Now, consider the case when $k+l+m+n\equiv 1 \pmod 2$. The change $k\to k+1$ reduces formula (4.1) to

$$s_{k+l+1}s_{m+n}w_{k-l+1}w_{m-n}+s_{k+n+1}s_{l+m}w_{k-n+1}w_{l-m}-s_{k+m+1}s_{l+n}w_{k-m+1}w_{l-n}=0, \tag{4.5}$$

where $k+l+m+n\equiv 0 \pmod 2$. The same change of variables (4.3) allows us to rewrite formula (4.5) as

$$s_{a+b+1}s_{a-b}w_{c+d+1}w_{c-d}+s_{a+d+1}s_{a-d}w_{b+c+1}w_{b-c}-s_{a+c+1}s_{a-c}w_{b+d+1}w_{b-d}=0. \tag{4.6}$$

Just as in the previous case, to complete the proof it remains to apply formulas (3.21) and (3.22) to all products of the form $s_{x+y+1}s_{x-y}$ and $w_{x+y+1}w_{x-y}$, respectively. □

Remark 4.2. Equality (4.1) can be rewritten as

$$D^{(0)}_{s,w}\begin{pmatrix} m_1, m_2\\ n_1, n_2\end{pmatrix}=s_{m_1+m_2}s_{n_1+n_2}w_{m_1-m_2}w_{n_1-n_2}. \tag{4.7}$$

Particular cases of this formula are given by relations (1.10).

Proposition 4.3. *For any integer or simultaneously half-integer m_1, m_2, m_3, n_1, n_2, and n_3,*

$$D^{(0)}_{s,w}\begin{pmatrix} m_1, m_2, m_3\\ n_1, n_2, n_3\end{pmatrix}=0.$$

In particular,

$$D^{(0)}_{w,w}\begin{pmatrix} m_1, m_2, m_3\\ n_1, n_2, n_3\end{pmatrix}=0.$$

Proof. Let us expand this determinant along the first row:

$$D^{(0)}_{s,w}\begin{pmatrix} m_1, m_2, m_3\\ n_1, n_2, n_3\end{pmatrix}=s_{m_1+n_1}w_{m_1-n_1}D^{(0)}_{s,w}\begin{pmatrix} m_2, m_3\\ n_2, n_3\end{pmatrix}-s_{m_1+n_2}w_{m_1-n_2}D^{(0)}_{s,w}\begin{pmatrix} m_2, m_3\\ n_1, n_3\end{pmatrix}$$
$$+s_{m_1+n_3}w_{m_1-n_3}D^{(0)}_{s,w}\begin{pmatrix} m_2, m_3\\ n_1, n_2\end{pmatrix}.$$

Applying formula (4.7) to the determinant of each 2×2 matrix, we arrive at the equality

$$D^{(0)}_{s,w}\begin{pmatrix}m_1,m_2,m_3\\ n_1,n_2,n_3\end{pmatrix}=s_{m_2+m_3}w_{m_2-m_3}\bigl(s_{m_1+n_1}w_{m_1-n_1}s_{n_2+n_3}w_{n_2-n_3}\\ -s_{m_1+n_2}w_{m_1-n_2}s_{n_1+n_3}w_{n_1-n_3}+s_{m_1+n_3}w_{m_1-n_3}s_{n_1+n_2}w_{n_1-n_2}\bigr).$$

The expression in parentheses vanishes by formula (4.1). $\square$

Remark 4.4. According to Proposition 4.3, the matrix $M^{(0,1)}_{s,w}$ has rank 2. Hence, all of its 4×4 submatrices have zero determinant. Therefore, equality (4.1) can be understood as the vanishing of the Pfaffian of the degenerate skew-symmetric matrix

$$M^{(0)}_{s,w}\begin{pmatrix}k,l,m,n\\ k,l,m,n\end{pmatrix}=\begin{pmatrix}0 & s_{k+l}w_{k-l} & s_{k+m}w_{k-m} & s_{k+n}w_{k-n}\\ s_{l+k}w_{l-k} & 0 & s_{l+m}w_{l-m} & s_{l+n}w_{l-n}\\ s_{m+k}w_{m-k} & s_{m+l}w_{m-l} & 0 & s_{m+n}w_{m-n}\\ s_{n+k}w_{n-k} & s_{n+l}w_{n-l} & s_{n+m}w_{n-m} & 0\end{pmatrix}.$$

Equalities (4.1) and (4.2) can be regarded as analogs of the Weierstrass three-term identity

$$\sigma(k+l)\sigma(k-l)\sigma(m+n)\sigma(m-n)+\sigma(k+n)\sigma(k-n)\sigma(l+m)\sigma(l-m)\\ -\sigma(k+m)\sigma(k-m)\sigma(l+n)\sigma(l-n)=0,$$

which can also be understood as the vanishing of the Pfaffian of the degenerate skew-symmetric matrix

$$\begin{pmatrix}0 & \sigma(k+l)\sigma(k-l) & \sigma(k+m)\sigma(k-m) & \sigma(k+n)\sigma(k-n)\\ \sigma(l+k)\sigma(l-k) & 0 & \sigma(l+m)\sigma(l-m) & \sigma(l+n)\sigma(l-n)\\ \sigma(m+k)\sigma(m-k) & \sigma(m+l)\sigma(m-l) & 0 & \sigma(m+n)\sigma(m-n)\\ \sigma(n+k)\sigma(n-k) & \sigma(n+l)\sigma(n-l) & \sigma(n+m)\sigma(n-m) & 0\end{pmatrix}.$$

Lemma 4.5. *Let a, b, and c be integers and $a+b+c=N$. Then*

$$s_as_bs_cw_Nw_{N-1}w_1^2=s_{N-1}s_0s_1w_1w_{N-a}w_{N-b}w_{N-c}\\ +s_N\bigl(s_0^2w_{a-1}w_{b-1}w_{c-1}-s_1s_{-1}w_aw_bw_cw_{N-1}\bigr). \tag{4.8}$$

Proof. Since the equalities $a+b=1$, $a+c=1$, and $b+c=1$ cannot hold simultaneously, we can assume in what follows that $a+b-1\neq 0$.

Substituting the values $k=(u-t)/2$, $n=-k$, $m=v+k$, and $l=u-k$ into (4.1), we obtain

$$s_us_vw_{u+v-t}w_t=s_{u+v-t}s_tw_uw_v-s_{u+v}s_0w_{u-t}w_{v-t}.$$

In particular, for $t=\pm1$,

$$s_us_vw_{u+v\mp1}w_{\pm1}=s_{u+v\mp1}s_{\pm1}w_uw_v-s_{u+v}s_0w_{u\mp1}w_{v\mp}.$$

Applying the above formula successively to the products $s_as_bw_{a+b-1}w_1$, $s_cs_{a+b-1}w_{a+b+c}w_{-1}$, and $s_cs_{a+b}w_{a+b+c-1}w_1$, we find

$$s_as_bs_cw_{a+b-1}w_{a+b+c}w_{a+b+c-1}w_1^2=s_cw_{a+b+c}w_{a+b+c-1}w_1\bigl(s_{a+b-1}s_1w_aw_b-s_{a+b}s_0w_{a-1}w_{b-1}\bigr)\\ =s_Nw_{a+b-1}\bigl(s_0^2w_{a-1}w_{b-1}w_{c-1}w_N-s_1s_{-1}w_aw_bw_cw_{N-1}\bigr)\\ +s_{N-1}w_{a+b}s_0s_1\bigl(w_aw_bw_{c+1}w_{N-1}-w_{a-1}w_{b-1}w_cw_N\bigr).$$

The expression in parentheses in the last row can be transformed with the help of (4.2) to the form $w_1 w_{a+c} w_{b+c} w_{a+b-1}$. Reducing the resulting equality by the factor $w_{a+b-1} \neq 0$, we arrive at equality (4.8). $\square$

Note that the same elliptic divisibility sequence $\{w_n\}$ may correspond to different Somos-4 sequences $\{s_n\}$ and $\{\widetilde{s}_n\}$. As pointed out above, this may happen if the sequences $\{s_n\}$ and $\{\widetilde{s}_n\}$ have the same set of parameters (α, β, T). An equivalent condition is that the same elliptic curve (2.8) corresponds to both sequences. Such pairs of sequences can also be viewed (see [4]) as solutions of the bilinear recurrence relations

$$s_{n+2}\widetilde{s}_{n-2} = \alpha_1 s_{n+1}\widetilde{s}_{n-1} + \beta_1 s_n \widetilde{s}_n, \qquad s_{n-2}\widetilde{s}_{n+2} = \alpha_2 s_{n-1}\widetilde{s}_{n+1} + \beta_2 s_n \widetilde{s}_n.$$

Theorem 4.6. *Suppose that one sequence $\{w_n\}$ corresponds to two Somos-4 sequences $\{s_n\}$ and $\{\widetilde{s}_n\}$. Then, for any integer or simultaneously half-integer k, l, m, x, y, and z, the determinant $D^{(0)}_{s,\widetilde{s}}\binom{k,l,m}{x,y,z}$ vanishes*:

$$\begin{vmatrix} s_{k+x}\widetilde{s}_{k-x} & s_{k+y}\widetilde{s}_{k-y} & s_{k+z}\widetilde{s}_{k-z} \\ s_{l+x}\widetilde{s}_{l-x} & s_{l+y}\widetilde{s}_{l-y} & s_{l+z}\widetilde{s}_{l-z} \\ s_{m+x}\widetilde{s}_{m-x} & s_{m+y}\widetilde{s}_{m-y} & s_{m+z}\widetilde{s}_{m-z} \end{vmatrix} = 0.$$

Proof. Assume that the number $N = k + l + m + x + y + z$ is different from zero and one. Otherwise, we can shift the numbering of the sequence $\{\widetilde{s}_n\}$ (pass to $\widetilde{s}'_n = \widetilde{s}_{n+t}$; see Remark 3.6) in such a way that this condition is indeed satisfied. Let us write the determinant $D^{(0)}_{s,\widetilde{s}}\binom{k,l,m}{x,y,z}$ as the sum

$$s_{k+x}s_{l+y}s_{m+z}\widetilde{s}_{k-x}\widetilde{s}_{l-y}\widetilde{s}_{m-z} + \dots$$

and multiply the resulting expression by $w_N^2 w_{N-1}^2 \neq 0$. Then we can apply formula (4.8) to each factor of the form $w_N w_{N-1} s_{k+x}s_{l+y}s_{m+z}$ and $w_N w_{N-1}\widetilde{s}_{k-x}\widetilde{s}_{l-y}\widetilde{s}_{m-z}$. In the expression obtained, we combine all terms containing the same powers of s_0, $s_{\pm 1}$, s_N, s_{N-1}, $\widetilde{s}_0$, $\widetilde{s}_{\pm 1}$, $\widetilde{s}_N$, and $\widetilde{s}_{N-1}$. Then each of the arising coefficients has the form $D^{(0)}_{w,w}\binom{\alpha_1,\alpha_2,\alpha_3}{\beta_1,\beta_2,,\beta_3}$ and vanishes by Proposition 4.3. For example, the coefficient of $s_N s_0^2 \widetilde{s}_N \widetilde{s}_0^2$ is equal to

$$w_{k+x-1}w_{l+y-1}w_{m+z-1}w_{k-x-1}w_{l-y-1}w_{m-z-1} + \dots = D^{(0)}_{w,w}\begin{pmatrix} k-1, l-1, m-1 \\ x, y, z \end{pmatrix} = 0.$$

The coefficient of $s_N s_0^2 \widetilde{s}_N \widetilde{s}_{-1}\widetilde{s}_1$ is

$$w_{k+x-1}w_{l+y-1}w_{m+z-1}w_{N-(k-x)}w_{N-(l-y)}w_{N-(m-z)} + \dots = D^{(0)}_{w,w}\begin{pmatrix} k-N_1, l-N_1, m-N_1 \\ x+N_1, y+N_1, z+N_1 \end{pmatrix} = 0,$$

where $N_1 = (N-1)/2$, and so on. $\square$

FUNDING

This work is supported by the Russian Science Foundation under grant 14-11-00335.

REFERENCES

1. J. Alman, C. Cuenca, and J. Huang, "Laurent phenomenon sequences," J. Algebr. Comb. **43** (3), 589–633 (2016).
2. M. O. Avdeeva and V. A. Bykovskii, "Hyperelliptic systems of sequences and functions," Dal'nevost. Mat. Zh. **16** (2), 115–122 (2016).
3. V. A. Bykovskii, "Hyperquasipolynomials and their applications," Funct. Anal. Appl. **50** (3), 193–203 (2016) [transl. from Funkts. Anal. Prilozh. **50** (3), 34–46 (2016)].

4. V. A. Bykovskii and A. V. Ustinov, “Somos-4 and elliptic systems of sequences,” Dokl. Math. **94** (3), 611–614 (2016) [transl. from Dokl. Akad. Nauk **471** (1), 7–10 (2016)].
5. Yu. N. Fedorov and A. N. W. Hone, “Sigma-function solution to the general Somos-6 recurrence via hyperelliptic Prym varieties,” J. Integrable Syst. **1** (1), xyw012 (2016).
6. S. Fomin and A. Zelevinsky, “The Laurent phenomenon,” Adv. Appl. Math. **28** (2), 119–144 (2002).
7. D. Gale, *Tracking the Automatic Ant and Other Mathematical Explorations: A Collection of Mathematical Entertainments Columns from the Mathematical Intelligencer* (Springer, New York, 1998).
8. A. N. W. Hone, “Elliptic curves and quadratic recurrence sequences,” Bull. London Math. Soc. **37** (2), 161–171 (2005).
9. A. N. W. Hone, “Sigma function solution of the initial value problem for Somos 5 sequences,” Trans. Am. Math. Soc. **359** (10), 5019–5034 (2007).
10. A. N. W. Hone, “Analytic solutions and integrability for bilinear recurrences of order six,” Appl. Anal. **89** (4), 473–492 (2010).
11. A. N. W. Hone and C. Swart, “Integrality and the Laurent phenomenon for Somos 4 and Somos 5 sequences,” Math. Proc. Cambridge Philos. Soc. **145** (1), 65–85 (2008).
12. A. A. Illarionov, “Functional equations and Weierstrass sigma-functions,” Funct. Anal. Appl. **50** (4), 281–290 (2016) [transl. from Funkts. Anal. Prilozh. **50** (4), 43–54 (2016)].
13. A. A. Illarionov, “Solution of functional equations related to elliptic functions,” Proc. Steklov Inst. Math. **299**, 96–108 (2017) [transl. from Tr. Mat. Inst. Steklova **299**, 105–117 (2017)].
14. A. A. Illarionov and M. A. Romanov, “On the connection between hyperelliptic systems of sequences and functions,” Dal’nevost. Mat. Zh. **17** (2), 210–220 (2017).
15. X. Ma, “Magic determinants of Somos sequences and theta functions,” Discrete Math. **310** (1), 1–5 (2010).
16. A. J. van der Poorten and C. S. Swart, “Recurrence relations for elliptic sequences: Every Somos 4 is a Somos k,” Bull. London Math. Soc. **38** (4), 546–554 (2006).
17. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Dover, New York, 1990).
18. R. Rochberg and L. A. Rubel, “A functional equation,” Indiana Univ. Math. J. **41** (2), 363–376 (1992).
19. M. Somos, “Step into the elliptic realm,” Preprint (2000), https://www.cut-the-knot.org/arithmetic/algebra/EllipticSeq.shtml
20. C. S. Swart, “Elliptic curves and related sequences,” PhD Thesis (R. Holloway and Bedford New Coll., Univ. London, Egham, 2003).
21. F. G. Tricomi, *Integral Equations* (Interscience, New York, 1957).
22. M. Ward, “Memoir on elliptic divisibility sequences,” Am. J. Math. **70** (1), 31–74 (1948).
23. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions* (Univ. Press, Cambridge, 1927).

Translated by I. Nikitin