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SHORT COMMUNICATIONS

On Somos-4 and Somos-5 Sequences

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A *Somos*-4 sequence is defined by the quadratic recurrence relation

$$\tau_{n+2}\tau_{n-2} = \alpha\tau_{n+1}\tau_{n-1} + \beta\tau_n^2 \tag{1}$$

with constant α and β. In the variables $f_n = \tau_{n-1}\tau_{n+1}\tau_n^{-2}$, Eq. (1) has the form

$$f_{n-1}f_n^2 f_{n+1} = \alpha f_n + \beta. \tag{2}$$

Finding a general solution of Eq. (1) (see [1]) is based on the fact that the sequence (2) has the first integral

$$J = f_n f_{n+1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n+1}}\right) + \frac{\beta}{f_n f_{n+1}}\,.$$

A *Somos*-5 sequence is defined by

$$\tau_{n+3}\tau_{n-2} = \widetilde{\alpha}\tau_{n+2}\tau_{n-1} + \widetilde{\beta}\tau_{n+1}\tau_n. \tag{3}$$

Theorem 1 (see [2]). *In the variables f_n, relation* (3) *defines the third-order recurrence sequence*

$$f_{n-1}f_n^2 f_{n+1}^2 f_{n+2} = \widetilde{\alpha} f_n f_{n+1} + \widetilde{\beta}, \tag{4}$$

which has the two independent first integrals

$$\widetilde{I} = f_{n-1}f_n f_{n+1} + \widetilde{\alpha}\left(\frac{1}{f_{n-1}} + \frac{1}{f_n} + \frac{1}{f_{n+1}}\right) + \frac{\beta}{f_{n-1}f_n f_{n+1}}\,, \tag{5}$$

$$\widetilde{J} = f_{n-1}f_n + f_n f_{n+1} + \alpha\left(\frac{1}{f_{n-1}f_n} + \frac{1}{f_n f_{n+1}}\right) + \frac{\beta}{f_{n-1}f_n^2 f_{n+1}}\,. \tag{6}$$

Any solution of Eq. (1) *satisfies Eq.* (3) *with parameters $\widetilde{\alpha} = -\beta$ and $\widetilde{\beta} = \alpha^2 + \beta J$, for which $\widetilde{J} = J$ and $\widetilde{I} = 2\alpha$.*

Theorem 2 (see [2]). *For any solution τ_n of the recurrence relation* (3)*, the subsequences $\tau_n^* = \tau_{2n}$ and $\tau_n^* = \tau_{2n+1}$ with even and odd numbers satisfy the Somos-4 equation*

$$\tau_{n+2}^*\tau_{n-2}^* = \alpha^*\tau_{n+1}^*\tau_{n-1}^* + \beta^*(\tau_n^*)^2 \tag{7}$$

in which $\alpha^* = \widetilde{\beta}^2$ and $\beta^* = \widetilde{\alpha}(\widetilde{\alpha}^3 + 2\widetilde{\beta}^2 + \widetilde{\alpha}\widetilde{\beta}\widetilde{J}\,)$.

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In the variables $h_n = f_{n+1}f_n = \tau_{n+2}\tau_{n-1}/(\tau_{n+1}\tau_n)$, Eqs. (4) and (6) have the form

$$h_{n-1}h_nh_{n+1} = \widetilde{\alpha}h_n + \widetilde{\beta}, \tag{8}$$

$$\widetilde{J} = h_{n-1} + h_n + \alpha\left(\frac{1}{h_{n-1}} + \frac{1}{h_n}\right) + \frac{\beta}{h_{n-1}h_n}\,. \tag{9}$$

The proof of the basic properties of the Somos-4 sequences is quite elementary; see [3] and [4]. The proof of Theorem 1 is elementary as well, while that of Theorem 2 uses an explicit expression for solutions of Eqs. (1) and (8) in terms of the Weierstrass function. However, Theorem 2 can also be proved without using elliptic functions. Such an elementary approach can be useful for studying more complicated sequences, for which no explicit formulas are known. The sequences most interesting in this respect are the Gale–Robinson sequences defined by (see [5])

$$\tau_n\tau_{n-k} = \alpha\tau_{n-l}\tau_{n-k+l} + \beta\tau_{n-m}\tau_{n-k+m}, \qquad 0 < l < m < k, \tag{10}$$

$$\tau_n\tau_{n-k} = \alpha\tau_{n-p}\tau_{n-k+p} + \beta\tau_{n-q}\tau_{n-k+q} + \gamma\tau_{n-r}\tau_{n-k+r}, \qquad 0 < p < q < r < k, \quad p+q+r = k. \tag{11}$$

These sequences are known to have the Laurent property (see [6]), and it is natural to expect that general solutions of the recurrence relations (10) and (11) can be described in terms of theta-functions of many variables.

Proof of Theorem 2. Let us transform the quantity

$$\frac{\tau_{n-4}\tau_{n+4}}{\tau_n^2} = h_{n-3}h_{n-2}h_{n-1}^2h_n^2h_{n+1}h_{n+2} = h_{n-1}h_n(\widetilde{\alpha}h_{n-2} + \widetilde{\beta}\,)(\widetilde{\alpha}h_{n+1} + \widetilde{\beta}\,).$$

Eliminating h_{n-2} and h_{n+1} by using the relations

$$h_{n-2} = \frac{\widetilde{\alpha}}{h_n} + \frac{\widetilde{\beta}}{h_{n-1}h_n} \qquad \text{and} \qquad h_{n+1} = \frac{\widetilde{\alpha}}{h_{n-1}} + \frac{\widetilde{\beta}}{h_{n-1}h_n}$$

and taking (9) into account, we obtain the following equation equivalent to (7):

$$\frac{\tau_{n-4}\tau_{n+4}}{\tau_n^2} = \widetilde{\beta}^2h_{n-1}h_n + \widetilde{\alpha}(\widetilde{\alpha}^3 + 2\widetilde{\beta}^2 + \widetilde{\alpha}\widetilde{\beta}\widetilde{J}\,) = \alpha^*\frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2} + \beta^*. \quad \square$$

We also give a criterion for a Somos-5 sequence to be a Somos-4 sequence.

Theorem 3. *A Somos-5 sequence defined by (3) and having first integrals $\widetilde{I}$ and $\widetilde{J}$ is a Somos-4 sequence if and only if $\widetilde{I}^2 = 4(\widetilde{\beta} + \widetilde{\alpha}\widetilde{J}\,)$.*

Proof. The necessity of the condition $\widetilde{I}^2 = 4(\widetilde{\beta} + \widetilde{\alpha}\widetilde{J}\,)$ follows from Theorem 1.

Let us prove its sufficiency. Note that

$$\widetilde{I}^2 - 4(\widetilde{\beta} + \widetilde{\alpha}\widetilde{J}\,) = \left(f_{n-1}f_nf_{n+1} + \widetilde{\alpha}\left(\frac{1}{f_n} - \frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right) - \frac{\beta}{f_{n-1}f_nf_{n+1}}\right)^2.$$

Therefore, the condition $\widetilde{I}^2 = 4(\widetilde{\beta} + \widetilde{\alpha}\widetilde{J}\,)$ means that

$$f_{n-1}f_nf_{n+1} + \widetilde{\alpha}\left(\frac{1}{f_n} - \frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right) - \frac{\beta}{f_{n-1}f_nf_{n+1}} = 0. \tag{12}$$

Relations (5) and (12) give the equation

$$f_{n-1}f_nf_{n+1} + \frac{\widetilde{\alpha}}{f_n} = \frac{\widetilde{I}}{2}\,,$$

which coincides with (2) for $\widetilde{\alpha} = -\beta$ and $\widetilde{I} = 2\alpha$. $\square$

Remark. In the general case, it follows from the proof of Theorem 3 that

$$f_{n-1}f_nf_{n+1}+\frac{\widetilde{\alpha}}{f_n}=\frac{\widetilde{I}\pm\widetilde{K}}{2},$$

where $\widetilde{K}=\sqrt{\widetilde{I}^2-4(\widetilde{\beta}+\widetilde{\alpha}\widetilde{J}\,)}$. The relation

$$\widetilde{I}=f_{n-1}f_nf_{n+1}+f_nf_{n+1}f_{n+2}+\widetilde{\alpha}\biggl(\frac{1}{f_n}+\frac{1}{f_{n+1}}\biggr)$$

implies that the signs in $\pm$ alternate, i.e.,

$$f_{n-1}f_nf_{n+1}+\frac{\widetilde{\alpha}}{f_n}=\frac{\widetilde{I}+(-1)^n\widetilde{K}}{2}$$

for $\widetilde{K}$ with the appropriately chosen sign. Thus, any Somos-5 sequence can be regarded as a Somos-4 sequence with the alternating coefficient

$$\alpha_n=\frac{\widetilde{I}+(-1)^n\widetilde{K}}{2}.$$

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