When algebraic entropy vanishes

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I. Overview

Probabilistic **DYNAMICS**

$\uparrow$

COMBINATORICS

$\uparrow$

Algebraic **DYNAMICS**

II. Rational maps

Projective space:
$\mathbf{CP}^n = (\mathbf{C}^{n+1} \setminus (0,0,\ldots,0)) \,/ \sim$, where $u \sim v$ iff $v = cu$ for some $c \neq 0$.

We write the equivalence class of $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbf{CP}^n$ as $(x_1 : x_2 : \ldots : x_{n+1})$.

The standard imbedding of affine n-space into projective n-space is
$(x_1, x_2, \ldots, x_n) \mapsto (x_1 : x_2 : \ldots : x_n : 1)$.

The "inverse map" is
$(x_1 : \ldots : x_n : x_{n+1}) \mapsto (\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$.

Geometrical version: $\mathbf{CP}^n$ is the set of lines through the origin in $(n+1)$-space.

The point $(a_1 : a_2 : \ldots : a_{n+1})$ in $\mathbf{CP}^n$ corresponds to the line $a_1x_1 = a_2x_2 = \cdots = a_{n+1}x_{n+1}$ in $\mathbf{C}^{n+1}$.

The intersection of this line with the hyperplane $x_{n+1} = 1$ is the point

$$(\frac{a_1}{a_{n+1}}, \frac{a_2}{a_{n+1}}, \ldots, \frac{a_n}{a_{n+1}}, 1)$$

(as long as $a_{n+1} \neq 0$).

We identity affine n-space with the hyperplane $x_{n+1} = 1$.

Affine n-space is a Zariski-dense subset of projective n-space.

A *rational map* is
a function from (a Zariski-dense subset of) $\mathbf{CP}^n$ to $\mathbf{CP}^m$ given by m rational functions of the affine coordinate variables,

or,

the associated function from a Zariski-dense subset of $\mathbf{C}^n$ to $\mathbf{CP}^m$
(e.g., the "function" $x \mapsto 1/x$ on affine 1-space, associated with the function $(x : y) \mapsto (y : x)$ on projective 1-space).

A *birational map* is a rational map f from $\mathbf{CP}^n$ to $\mathbf{CP}^n$ with a rational inverse g (satisfying $f \circ g = g \circ f$ on a Zariski-dense subset of $\mathbf{CP}^n$).

Example:
Affine map: $(x, y) \mapsto (xy, y)$
(with inverse $(x, y) \mapsto (x/y, y)$)
Projective map: $f : (x : y : z) \mapsto (xy : yz : z^2)$
(with inverse $g : (x : y : z) \mapsto (xz : y^2 : yz)$).

Check:
$f(g(x : y : z)) = ((xz)(y^2) : (y^2)(yz) : (yz)^2) = (xy^2z : y^3z : y^2z^2) = (x : y : z)$.

Every rational map from $\mathbf{CP}^n$ to $\mathbf{CP}^m$ can be written in the form $(x_1 : \ldots : x_{n+1}) \mapsto (p_1(x_1, \ldots, x_{n+1}) : \ldots : p_{m+1}(x_1, \ldots, x_{n+1}))$ where the $m+1$ polynomials $p_1, \ldots, p_{m+1}$ are homogeneous polynomials of the same degree (call it d) having no joint common factor.

We call d the *degree* of the map.

E.g., the map $f : (x : y : z) \mapsto (xy : yz : z^2)$ is of degree 2, and the composite map $f^2 = f \circ f$ is of degree 3: $((xy)(yz) : (yz)(z^2) : (z^2)^2) = (xy^2z : yz^3 : z^4) = (xy^2 : yz^2 : z^3)$.

The degree of a map $f : \mathbf{CP}^n \to \mathbf{CP}^n$ is also equal to the number of intersections between the forward image (under f) of a generic line in $\mathbf{CP}^n$ and a generic hyperplane in $\mathbf{CP}^n$.

III. Algebraic entropy (Bellon-Viallet)

Fact: For all rational maps f, g,

$$\deg(f \circ g) \leq \deg(f)\deg(g).$$

Consequence #1: By sub-additivity,

$$\frac{1}{N} \log \deg(f^{(N)})$$

converges to a limit, called the (Bellon-Viallet) **algebraic entropy** of f.

Consequence #2: If $g = \phi^{-1} \circ f \circ \phi$ for some birational ϕ, then f and g have the same algebraic entropy.

That is, algebraic entropy is invariant under birational conjugacy.

(See [BV] for more details.)

Example: $(x,y) \mapsto (y,xy) \mapsto (xy,xy^2) \mapsto (xy^2,x^2y^3) \mapsto (x^2y^3,x^3y^5) \mapsto (x^3y^5,x^5y^8) \mapsto \ldots$ has algebraic entropy $\frac{1+\sqrt{5}}{2}$.

More generally, if $A = (a_{i,j})_{i,j=1}^n$ is a toral endomorphism, then the algebraic entropy of the map

$(x_1, x_2, \ldots) \mapsto (x_1^{a_{1,1}} x_2^{a_{1,2}} \cdots, x_1^{a_{2,1}} x_2^{a_{2,2}} \cdots, \ldots)$

is the spectral radius of A.

(Is there some variant of Bellon and Viallet's algebraic definition that agrees with topological entropy in the case of toral endomorphisms?)

The Hénon map $(x, y) \mapsto (1 + y - ax^2, bx)$ projectivizes as
$(x : y : t) \mapsto (t^2 + ty - ax^2 : btx : t^2)$.

For any non-zero constants a and b, the Nth iterate of this map has degree 2^N, so every non-degenerate Hénon map has algebraic entropy $\log 2$.

The degree of a map $f^{(N)}$ is equal to the number of intersections between the forward image (under $f^{(N)}$) of a generic line in $\mathbf{CP}^n$ and a generic hyperplane in $\mathbf{CP}^n$.

Thus algebraic entropy concerns the growth rate of the number of intersections between one submanifold and the image of another submanifold, and is related to the intersection-complexity research program of Arnold (see [A1] and [A2]).

Viallet's conjecture: For any rational map f, the sequence $(\deg(f^{(N)}))_{N=1}^{\infty}$ satisfies some linear recurrence with constant coefficients and leading coefficient 1. Consequently, the algebraic entropy of a rational map is always the logarithm of an algebraic integer.

Late-breaking news (10/28/03): The first half of this conjecture appears to be false: the (multiplicative) toral automorphism

$$(x, y, z) \mapsto (y/x, z/x, x)$$

has a degree sequence that does not satisfy any linear recurrence of degree less than 25.

IV. Zero entropy

The generic rational map of degree d has algebraic entropy $\log d$; to get the entropy to be smaller, "gratuitous cancellations" must occur. When algebraic entropy vanishes, so much cancellation occurs that the degrees of the iterates of the map grow sub-exponentially.

An extreme case is when the sequence of iterates of f is periodic.

Example: $(x, y) \mapsto (y, (y+1)/x)$ has period 5.

A less degenerate case is

$$f : (x, y) \mapsto (y, (y^2 + 1)/x)$$

with $\deg(f^{(N)}) = 2N$; $\frac{1}{N} \log 2N \to 0$, so this f has algebraic entropy 0.

This f also possesses the *Laurent property* [FZ]: there exist polynomials $p_N(x, y)$ and $q_N(x, y)$ and monomials $m_N(x, y)$ such that

$$f^{(N)}(x, y) = \left(\frac{p_N(x, y)}{m_N(x, y)}, \frac{q_N(x, y)}{m_N(x, y)}\right).$$

A *Laurent polynomial* in x, y is:

an element of $\mathbf{C}[x, y]$ divided by some monomial $x^i y^j$;

or equivalently,

a sum of *Laurent monomials* $cx^i y^j$ with $c \in \mathbf{C}$ and $i, j \in \mathbf{Z}$ (positive, negative or zero);

or equivalently,

an element of $\mathbf{C}[x, x^{-1}, y, y^{-1}]$.

E.g., $\dfrac{y^2+1}{x} = x^{-1}y^2 + x^{-1}y^0$.

Fomin and Zelevinsky have studied the composite maps $f_b \circ f_c$ where $f_b(x,y) = (y, (y^b+1)/x)$ and $f_c(x,y) = (y, (y^c+1)/x)$.
These composite maps are related to Lie algebras and Kac-Moody Lie algebras of rank 2.

Fomin and Zelevinsky developed a theory of "cluster algebras" [FZ1] and have used it to show that compositions of the map $f_b \circ f_c$ are Laurent.

They observed that all the coefficients of these Laurent polynomials appear to be positive integers.

This has been proved for the cases $bc \leq 4$ ([FZ2], [SZ], [MP]).

V. A one-dimensional SFT

The special case $b = c = 2$ relates to the golden mean shift.
We lift the map $f_2 : (x, y) \mapsto (y, (y^2 + 1)/x)$ to an "algebraic cellular automaton"

$$F_2 : (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{y}, \mathbf{z}),$$

where

$$\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots),$$
$$\mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots),$$

and

$$\mathbf{z} = (\ldots, z_{-1}, z_0, z_1, \ldots)$$

with

$$z_n = (y_{n-1} y_{n+1} + 1)/x_n.$$

Note that if the sequences $\mathbf{x}$ and $\mathbf{y}$ are constant (i.e., $x_n = x_0$ and $y_n = y_0$ for all n) then $\mathbf{z}$ is also constant, with $z_0 = (y_0^2 + 1)/x_0$, which is our original map f_2.

$$
\begin{aligned}
[F_2(\mathbf{x},\mathbf{y})_2]_0 &= y_{-1}y_1x_0^{-1} + x_0^{-1} \\
[F_2^{(2)}(\mathbf{x},\mathbf{y})_2]_0 &= \text{a sum of 5 Laurent monomials} \\
[F_2^{(3)}(\mathbf{x},\mathbf{y})_2]_0 &= \text{a sum of 13 Laurent monomials} \\
[F_2^{(4)}(\mathbf{x},\mathbf{y})_2]_0 &= \text{a sum of 34 Laurent monomials}
\end{aligned}
$$

etc.

All these Laurent monomials have coefficient $+1$.

Fact: Every entry in $F_2^{(N)}(\mathbf{x}, \mathbf{y})_2$ is a Laurent polynomial in the variables x_n (with $|n| < N$ and $n \not\equiv N \bmod 2$) and the variables y_n (with $|n| \leq N$ and $n \equiv N \bmod 2$) in which all exponents are between -1 and $+1$ and all coefficients are equal to 1. Moreover, there is a simple bijection between the Laurent monomials that occur and the domino tilings of the rectangle $[-N, +N] \times [-1, +1]$ in $\mathbf{R}^2$.

Example ($N = 2$):

```
  o---o---o---o---o
  |       |       |          1   -1   1    -1   1
--4---2---4---2---4--      y    x    y    x    y
  |       |       |         -2   -1   0    1    2
  o---o---o---o---o
```

```
  o---o---o---o---o
  |       |   |   |         1    -1   0    -1   0
--4---2---3   2   3--      y    x    y    x    y
  |       |   |   |         -2   -1   0    1    2
  o---o---o---o---o
```

```
  o---o---o---o---o
  |   |   |       |         0    -1   0    -1   1
--3   2   3---2---4--      y    x    y    x    y
  |   |   |       |         -2   -1   0    1    2
  o---o---o---o---o
```

```
  o---o---o---o---o
  |   |   |   |   |         0    -1   -1   -1   0
--3   2   2   2   3--      y    x    y    x    y
  |   |   |   |   |         -2   -1   0    1    2
  o---o---o---o---o
```

```
  o---o---o---o---o
  |   |       |   |         0    0    -1   0    0
--3   3---2---3   3--      y    x    y    x    y
  |   |       |   |         -2   -1   0    1    2
  o---o---o---o---o
```

Each numerical marking in the tiling tells how many tile-edges meet at the marked vertex (where we add two notional tile-edges at the left and right for convenience).
Each exponent in the Laurent polynomial is 3 less than the corresponding vertex-marking.

Furthermore, each domino-tiling of the 2-by-$2N$ rectangle can be coded by a sequence of $2N-1$ bits, subject to the rule that no two 1's can appear in a row: we put 0's along vertical break-lines in the tiling, 1's everywhere else.

```
  o---o---o---o---o
  |       |       |
--o---1---0---1---o--     1   0   1
  |       |       |
  o---o---o---o---o

  o---o---o---o---o
  |       |   |   |
--o---1---0   0   o--     1   0   0
  |       |   |   |
  o---o---o---o---o

  o---o---o---o---o
  |   |   |       |
--o   0   0---1---o--     0   0   1
  |   |   |       |
  o---o---o---o---o

  o---o---o---o---o
  |   |   |   |   |
--o   0   0   0   o--     0   0   0
  |   |   |   |   |
  o---o---o---o---o

  o---o---o---o---o
  |   |       |   |
--o   0---1---0   o--     0   1   0
  |   |       |   |
  o---o---o---o---o
```

So the numerator of $[F_2^{(N)}(\mathbf{x},\mathbf{y})_2]_0$ is a sum of monomials, which individually represent the different words of length $2N-1$ in the golden mean shift.

We can think of the numerator as representing the uniform distribution on the words of length $2N-1$.

If we take a weak limit of this distribution as the rectangle gets longer and longer and the ends go off to infinity, we get the measure of maximal entropy for the golden mean shift.

This works even if we look at the shifted distribution in the vicinity of location rN for any r with $-1 < r < 1$.

If we modify F_2 by using the rule

$$z_n = (y_{n-1}y_{n+1} - 1)/x_n$$

instead of the rule

$$z_n = (y_{n-1}y_{n+1} + 1)/x_n$$

and we put $x_n = 1$ for all n, then $[F_2^{(N)}(\mathbf{x}, \mathbf{y})_2]_0$ is just the determinant of the $(N+1)$-by-$(N+1)$ tridiagonal matrix with $y_{-N}, y_{-N+2}, \ldots, y_N$ on the main diagonal and 1's on the immediately adjacent diagonals above and below:

$$\begin{pmatrix} y_{-N} & 1 & 0 & 0 & \ldots & 0 \\ 1 & y_{-N+2} & 1 & 0 & \ldots & 0 \\ 0 & 1 & y_{-N+4} & 1 & \ldots & 0 \\ 0 & 0 & 1 & y_{-N+6} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & y_N \end{pmatrix}$$

VI. A two-dimensional SFT

The octahedron recurrence (Robbins): Given $\mathbf{x} = (x_{i,j})_{i,j=-\infty}^{\infty}$ and $\mathbf{y} = (y_{i,j})_{i,j=-\infty}^{\infty}$, put $D(\mathbf{x},\mathbf{y}) = (\mathbf{y},\mathbf{z})$ where

$$z_{i,j} = \frac{y_{i-1,j}y_{i+1,j} + \lambda y_{i,j-1}y_{i,j+1}}{x_{i,j}}$$

(for some fixed λ). See [RR].

$$[F_2(\mathbf{x},\mathbf{y})_2]_{0,0} = \text{a sum of 2 Laurent monomials}$$

$$[F_2^{(2)}(\mathbf{x},\mathbf{y})_2]_{0,0} = \text{a sum of 8 Laurent monomials}$$

$$[F_2^{(3)}(\mathbf{x},\mathbf{y})_2]_{0,0} = \text{a sum of 64 Laurent monomials}$$

$$[F_2^{(4)}(\mathbf{x},\mathbf{y})_2]_{0,0} = \text{a sum of 1024 Laurent monomials}$$

etc.

All these Laurent monomials have coefficient $+1$.

Fact [EKLP]: Every entry in $D_2^{(N)}(\mathbf{x},\mathbf{y})$ is a Laurent polynomial in the variables $x_{i,j}$ (with $|i|+|j|<N$ and $i+j\not\equiv N \bmod 2$) and the variables $y_{i,j}$ (with $|i|+|j|\le N$ and $i+j\equiv N \bmod 2$) in which all exponents are between -1 and $+1$ in which all coefficients are equal to a power of λ.

There is a simple bijection between the Laurent monomials that occur and the domino tilings of the region

$$\begin{array}{rrcl} & [-N,N] & \times & [-1,1] \\ \cup & [-(N-1),N-1] & \times & [-2,2] \\ \cup & [-(N-2),N-2] & \times & [-3,3] \\ & & \cdots & \\ \cup & [-1,1] & \times & [-N,N] \end{array}$$

in $\mathbf{R}^2$, called the *Aztec diamond* of order N.

Example ($N = 2$): The tiling

```
          |
      o---3---o
      |       |
  o---3---3---4---o
  |       |   |   |
--4---2---3   2   3--
  |       |   |   |
  o---3---3---4---o
      |       |
      o---3---o
          |
```

is associated with the monomial

$$y^0_{0,2}$$

$$y^0_{-1,1}\ x^0_{0,1}\ y^1_{1,1}$$

$$y^1_{-2,0}\ x^{-1}_{-1,0}\ y^0_{0,0}\ x^{-1}_{1,0}\ y^0_{2,0}$$

$$y^0_{-1,-1}\ x^0_{0,-1}\ y^1_{1,-1}$$

$$y^0_{0,-2}$$

times a power of λ (specifically λ^1).

As in the preceding section, each numerical marking in the tiling tells how many tile-edges meet at the marked vertex (where we add four notional tile-edges at the four corners), and each exponent in the Laurent polynomial is 3 less than the corresponding vertex-marking.

$[D^{(N)}{}_2]_{0,0}$, viewed as a mapping from $\mathbf{C}^{N^2+(N+1)^2}$ to $\mathbf{C}$, has degree $\leq 2(N^2 + (N+1)^2)$. Ordinarily we would expect exponential growth, so in some sense we are again in a situation where algebraic entropy vanishes (though we have not defined the algebraic entropy of an algebraic cellular automaton).

If we specialize to $\lambda = -1$ and we put $x_{i,j} = 1$ for all i, j, then $[D^{(N)}{}_2]_{0,0}$ is just the determinant of the $(N+1)$-by-$(N+1)$ matrix whose entries are the $y_{i,j}$ with $i + j \equiv N \pmod 2$ and $|i| + |j| \leq N$.
(This is Dodgson's algorithm for computing determinants via "condensation".)
Dodgson condensation is what motivated David Robbins and his collaborators to invent the λ-determinant in the first place.
If we specialize to $\lambda = +1$ and we put $x_{i,j} = 1$ for all i, j, then the terms of $[D^{(N)}{}_2]_{0,0}$ correspond to the "alternating-sign matrices" (ASMs) of order $N + 1$. Counting these was the most notorious open problem in enumerative combinatorics of the 1980s (eventually solved by Zeilberger in the 1990s). For an account of this story, see [BrP].

VII. Multiple Gibbs measures

There is a unique measure of maximal entropy for the $\mathbf{Z}^2$-action supported on the set of domino tilings of the plane (proof by Burton and Pemantle [BuP], corrected by Lyons [L]).

If one takes the weak limit near the origin of the uniform measure on domino tilings of the Aztec diamond of order N, then it is believed that one gets the Burton-Pemantle measure.

But this is demonstrably false if one looks at the shifted distribution in the vicinity of location (rN, sN) for any $(r, s) \neq (0, 0)$. (See [CEP].)

Take (r,s) with $|r|+|s| \leq 1$. It is known that if $r^2+s^2 \geq \frac{1}{2}$, then the distribution in the vicinity of location (rN, sN) converges to a trivial probability distribution (the "brick-wall tiling"), whereas if $r^2+s^2 < \frac{1}{2}$, this is not the case. (This is the "arctic circle theorem": see [JPS].)

There exist functions $E_{r,s}$ on the set of domino tilings of the plane, for which there are believed to be unique measure of maximal pressure $\mu_{r,s}$, such that it is believed that $\mu_{r,s}$ is the weak-limit of the distributions in the vicinity of location (rN, sN) given by uniform measure of Aztec diamonds of order N as N goes to infinity. (See [CKP].)

These measures are "conditionally uniform": if we restrict the measure to any finite set of tilings (which all agree outside of some bounded region), it becomes the uniform measure on that finite set of tilings.

For mixing SFTs in 1 dimension, the only conditionally uniform measure is the measure of maximal entropy. In 2 dimensions, we can have uncountably many such measures. The hope is to classify them. For domino tilings of the plane, it appears that the conditionally uniform (translation-invariant, ergodic) measures of positive entropy can be parametrized by r and s.

That is, the conditionally uniform measures of positive entropy appear to be exactly the different sorts of local behavior that are visible in different locations in the randomly tiled Aztec diamond.

There are similar effects for random rhombus-tilings of the triangulated plane (see [CLP]).

In this case, the positive-entropy conditionally uniform measures can be provably parametrized by two parameters that specify the local densities of the three orientations of tile, or equivalently, specify the local tilt of the height-representation (where we interpret the tiles and square faces of a polyhedral surface, viewed obliquely). See the (soon-to-be-available) preprint of Kenyon, Okounkov and Sheffield [KOS].

VIII. Three other examples

A. Somos sequences

The *Somos-4 recurrence* is

$$S_n S_{n-4} - S_{n-1} S_{n-3} - S_{n-2} S_{n-2} = 0,$$

i.e.,

$$S_n = \frac{S_{n-1} S_{n-3} + S_{n-2} S_{n-2}}{S_{n-4}}.$$

This corresponds to the map

$$(w, x, y, z) \mapsto (x, y, z, (xz + yy)/w).$$

The associated degree-2 projective map

$$(w : x : y : z : t) \mapsto (tx : ty : tz : xz + yy : t^2)$$

from $\mathbf{CP}^4$ to $\mathbf{CP}^4$ has iterates of degrees

$$1, 2, 3, 5, 8, 10, 14, 18, 22, 28, 33, 39, 46, 52, \ldots$$

growing sub-exponentially (more specifically, quadratically).

One can lift this 1-dimensional recurrence to the 2-dimensional algebraic cellular automaton

$$S_{n,i,j} = \frac{S_{n-1,i-1,j}S_{n-3,i+1,j} + S_{n-2,i,j-1}S_{n-2,i,j+1}}{S_{n-4,i,j}}.$$

Like the octahedron recurrence, this recurrence has the Laurent property, and one can show that the resulting Laurent polynomials have all coefficients equal to $+1$. It also has the property that the degree-growth is sub-exponential.
These Laurent polynomials encode a combinatorial interpretation of the integer sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, \ldots$$

obtained by putting $S(0) = S(1) = S(2) = S(3) = 1$ and defining all $S(n)$ with $n > 3$ by the Somos recurence. (See [G], [BPW], [S].)

B. The cube recurrence

The two-dimensional algebraic cellular automaton $S_{i,j,k} =$

$$\frac{S_{i-1,j,k}S_{i,j-1,k-1} + S_{i,j-1,k}S_{i-1,j,k-1} + S_{i-1,j,k}S_{i,j-1,k-1}}{S_{i-1,j-1,k-1}}$$

also exhibits sub-exponential degree-growth.

If one sets $S_{i,j,k}$ equal to formal indeterminates for $-1 \leq i + j + k \leq 1$, all the coefficients of the resulting Laurent polynomials are equal to 1. The monomials are in bijection with a new kind of combinatorial objected, devised by Carroll and Speyer: *groves*. (See [CS].)

C. Fortresses

The study of the discrete Painlevé II recurrence

$$x_n + (1 - (x_n)^2)(x_{n+1} + x_{n-1}) = 0$$

leads one to study the linked recurrences

$$P_{n+1,i,j} = \frac{P_{n,i-1,j}P_{n,i+1,j} + Q_{n,i,j-1}Q_{n,i,j+1}}{P_{n-2,i,j}}$$

$$Q_{n+1,i,j} = \frac{Q_{n,i,j}Q_{n,i,j} + P_{n,i,j}P_{n,i,j}}{Q_{n-2,i,j}}$$

If one puts $P_{n,i,j} = x_{i,j}$ for $n = 0$ or 1 and $Q_{n,i,j} = y_{i,j}$ for $n = 0$ or -1, then all the $P_{n,i,j}$'s and $Q_{n,i,j}$'s become Laurent polynomials with all coefficients equal to 1.

These Laurent polynomials are representations of "diabolo tilings of fortresses" [Y].

(The preceding phase diagram is taken from [K].)

IX. Summary

What's wanted is a dynamical systems theory in the category of algebraic geometry and rational/birational maps.
This will supplement current efforts by researchers in integrable systems who have their own set of tools.
It will also provide interesting new sorts of combinatorial objects.

Moral:

When algebraic entropy vanishes,

there's often a combinatorial reason.

X. References

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See also links accessible from the following web-pages:

`www.math.wisc.edu/~propp/somos.html`
`www.math.wisc.edu/~propp/bilinear.html`
`www.math.wisc.edu/~propp/reach/`