Domino tilings with barriers

In memory of Gian-Carlo Rota

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In this paper, we continue the study of domino-tilings of Aztec diamonds (introduced in [1] and [2]). In particular, we look at certain ways of placing "barriers" in the Aztec diamond, with the constraint that no domino may cross a barrier. Remarkably, the number of constrained tilings is independent of the placement of the barriers. We do not know of a simple combinatorial explanation of this fact; our proof uses the Jacobi-Trudi identity.

I. Statement of result.

An **Aztec diamond of order** n is a region composed of $2n(n+1)$ unit squares, arranged in bilaterally symmetric fashion as a stack of $2n$ rows of squares, the rows having lengths

$$2, 4, 6, ..., 2n-2, 2n, 2n, 2n-2, ..., 6, 4, 2.$$

A **domino** is a 1-by-2 (or 2-by-1) rectangle. It was shown in [1] that the Aztec diamond of order n can be tiled by dominoes in exactly $2^{n(n+1)/2}$ ways.

Here we study **barriers**, indicated by darkened edges of the square grid associated with an Aztec diamond. These are edges that no domino is permitted to cross. (If one prefers to think of a domino tiling of a region as a perfect matching of a dual graph whose vertices correspond to grid-squares and whose edges correspond to pairs of grid-squares having a shared edge, then putting down a barrier in the tiling is tantamount to removing an edge from the dual graph.) Figure 1(a) shows an Aztec diamond of order 8 with barriers, and Figure 1(b) shows a domino-tiling that is compatible with this placement of barriers.

The barrier-configuration of Figure 1(a) has special structure. Imagine a line of slope 1 running through the center of the Aztec diamond (the "spine"), passing through $2k$ grid-squares, with $k = \lceil n/2 \rceil$. Number these squares from lower left (or "southwest") to upper right (or "northeast") as squares 1 through $2k$. For each such square, we may place barriers on its bottom and right edges (a "zig"), barriers on its left and top edges (a "zag"), or no barriers at all ("zip"). Thus Figure 1 corresponds to the sequence of decisions "zip, zig, zip, zag, zip, zag, zip, zig." Notice that in this example, for all i, the ith square has a zig or a zag if i is even and zip if i is odd. Henceforth (and in particular in the statement of the following Theorem) we assume that the placement of the barriers has this special form.

Theorem 1: Given a placement of barriers in the Aztec diamond as described above, the number of domino-tilings compatible with this placement is $2^{n(n+1)/2}/2^k$.

Some remarks on the Theorem:

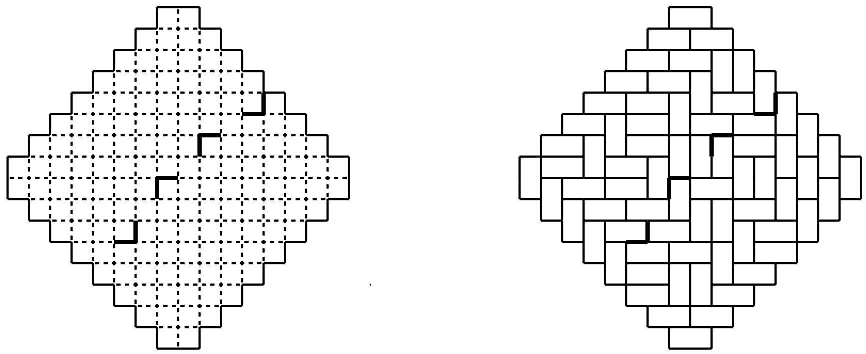
(1) The formula for the number of tilings makes no mention at all of the particular pattern of zigs and zags manifested by the barriers. Since there are k even-indexed squares along the spine, there are 2^k different barrier-configurations, all of which are claimed to have equal numbers of compatible tilings.

(2) Each domino-tiling of the Aztec diamond is compatible with exactly one barrier configuration (this will be explained more fully in section II). Hence, summing the formula in the Theorem over all barriers configurations, one gets $2^k \cdot 2^{n(n+1)/2}/2^k$, which is $2^{n(n+1)/2}$, the total number of tilings.

(3) 180-degree rotation of the Aztec diamond switches the odd-indexed and even-indexed squares along the spine, so the Theorem remains true if we consider barrier-configurations in which the ith square has a zig or a zag if i is odd and zip if i is even.

II. Preliminaries for the proof.

Consider a particular tiling of an Aztec diamond, and consider a particular square along the spine. If that square shares a domino with the square to its left, or above it, then placing a zag at that square is incompatible with the tiling. On the other hand, if the square shares a domino with the square to its right, or below it, then placing a zig at that square is incompatible with the tiling. It follows that for each domino-tiling, there is a unique compatible way of placing zigs and zags along the spine. This holds true whether one only puts zigs and zags at every other location along the spine (as in Figure 1(a)) or at every location along the spine. In the case of the tiling depicted



(a) (b)

FIG. 1. Barriers and tiling.

in Figure 1(b), the full sequence of zigs and zags goes "zag, zig, zig, zag, zig, zag, zag, zig."

Each such sequence must contain equal numbers of zigs and zags. For, suppose we color the unit squares underlying the Aztec diamond in checkerboard fashion, so that the squares along the spine are white and so that each white square has four only neighbors (and vice versa). The barriers divide the Aztec diamond into two parts, each of which must have equal numbers of black and white squares (since each part can be tiled by dominoes). It follows that the white squares along the spine must be shared equally by the part northwest of the diagonal and the part southeast of the diagonal.

Given a sequence of k zigs and k zags, let $1 \leq a_1 < a_2 < ... < a_k \leq 2k$ be the sequence of locations of the zigs, and let $1 \leq b_1 < b_2 < ... < b_k \leq 2k$ be the sequence of locations of the zags. Note that the sets $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_k\}$ are disjoint with union $\{1, 2, ..., 2k\}$. Let us call them a **balanced (ordered) partition** of $\{1, 2, ..., 2k\}$. It is proved in section 4 of [1] that the number of compatible domino-tilings of the Aztec diamond of order n is

$$\left(\prod_{1 \leq i < j \leq k} \frac{a_j - a_i}{j - i} \right) \left(\prod_{1 \leq i < j \leq k} \frac{b_j - b_i}{j - i} \right) 2^{k'(k'+1)} \quad (1)$$

where $k' = \lfloor n/2 \rfloor$. (This is equivalent to Theorem 2 in [4].) For instance, the tiling shown in Figure 1(b) determines the balanced partition $A = \{2, 3, 5, 8\}$, $B = \{1, 4, 6, 7\}$, and there are 2025 compatible tilings.

If we sum (1) over all balanced partitions of $\{1, 2, ..., 2k\}$ we must of course get $2^{n(n+1)/2}$. Theorem 1 claims that if we sum (1) over only those balanced partitions A, B which have certain specified even numbers in A (and the remaining even numbers in B), we get $2^{n(n+1)/2}/2^k$. Thus, to prove Theorem 1, it suffices to prove that

$$\sum_{(A,B)} \left(\prod_{1 \leq i < j \leq k} \frac{a_j - a_i}{j - i} \right) \left(\prod_{1 \leq i < j \leq k} \frac{b_j - b_i}{j - i} \right) \quad (2)$$

is independent of $A^* \subseteq \{2, 4, ..., 2k\}$, where the (A, B) in the sum ranges over all balanced partitions of $\{1, 2, ..., 2k\}$ such that $A \cap \{2, 4, ..., 2k\} = A^*$. Note that in this formulation, n has disappeared from the statement of the result, as has the Aztec diamond itself.

III. Restatement in terms of determinants.

We can interpret the left-hand side of (2) using Schur functions and apply the Jacobi-Trudi identity. The expression

$$\prod_{1 \leq i < j \leq k} \frac{a_j - a_i}{j - i} \quad (3)$$

is equal to the number of semistandard Young tableaux of shape $\lambda = (a_k - k, a_{k-1} - (k-1), ..., a_2 - 2, a_1 - 1)$ with parts at most k. That is to say, if one forms an array of unit squares forming left-justified rows of lengths $a_k - k, ..., a_1 - 1$ (from top to bottom), (3) gives the numbers of ways of filling in the boxes with numbers between 1 and k so that entries are weakly increasing from left to right and strictly increasing from top to bottom.

For background information on Young tableaux, Schur functions, and the Jacobi-Trudi identity, see [5], [6], [7], or [8]. In particular, for the definition of Schur functions and a statement of the Jacobi-Trudi identity, see formulas (5.13) and (3.4) of [5], Definition 4.4.1 and Theorem 4.5.1 of [6], or Definition 7.5.1 and Theorem 7.11.1 of [8].

If we associate with each semistandard Young tableau the monomial

$${x_1}^{m_1} {x_2}^{m_2} \cdots {x_k}^{m_k}$$

where m_i is the number of entries equal to i in the tableau, then the sum of the monomials associated with the tableau is the Schur function $s_\lambda(x_1, x_2, ..., x_k, 0, 0, ...)$. By the Jacobi-Trudi identity, this is equal to the determinant

$$\begin{vmatrix} h_{a_k-k} & \cdots & h_{a_k-2} & h_{a_k-1} \\ h_{a_{k-1}-k} & \cdots & h_{a_{k-1}-2} & h_{a_{k-1}-1} \\ \vdots & & \vdots & \vdots \\ h_{a_1-k} & \cdots & h_{a_1-2} & h_{a_1-1} \end{vmatrix}$$

where h_m is 0 if $m < 0$ and otherwise is equal to the sum of all monomials in $x_1, ..., x_k$ with total degree m (so that $h_0 = 1$, $h_1 = x_1 + ... + x_k$, $h_2 = x_1^2 + x_1x_2... + x_k^2$, etc.).

Thus, if we let $v(m)$ denote the length-k row-vector

$$(h_{m-k}, \cdots, h_{m-2}, h_{m-1}),$$

we see that the summand in (2) is the determinantal product

$$\begin{vmatrix} v(a_k) \\ \vdots \\ v(a_1) \end{vmatrix} \cdot \begin{vmatrix} v(b_k) \\ \vdots \\ v(b_1) \end{vmatrix}$$

specialized to $x_1 = x_2 = ... = x_k = 1$. To prove Theorem 1, it will suffice to show that this product, summed over all balanced partitions (A, B) with $A \cap \{2, 4, ..., 2k\} = A^*$, yields

$$\begin{vmatrix} v(2k) \\ v(2k-2) \\ \vdots \\ v(2) \end{vmatrix} \cdot \begin{vmatrix} v(2k-1) \\ v(2k-3) \\ \vdots \\ v(1) \end{vmatrix}.$$

For, since this expression is independent of A^*, and since the sum of this expression over all 2^k possible values of $A^* \subseteq \{2, ..., 2k\}$ is $2^{n(n+1)/2}$ (by the result proved in [1]), the value of the expression must be $2^{n(n+1)/2-k}$, as claimed in Theorem 1.

It is interesting to note that one can also evaluate the preceding determinantal product directly. Appealing to the Jacobi-Trudi identity, we see that the product is

$$s_\sigma(x_1, ..., x_k)s_\tau(x_1, ..., x_k)$$

where $\sigma = (k, k-1, ..., 1)$ and $\tau = (k-1, k-2, ..., 0)$. It is known that

$$s_\sigma(x_1, ..., x_k) = x_1 \cdots x_k \prod_{i<j}(x_i + x_j)$$

and

$$s_\tau(x_1, ..., x_k) = \prod_{i<j}(x_i + x_j) \ ,$$

so that the determinantal product is

$$x_1 \cdots x_k \prod_{i<j}(x_i + x_j)^2.$$

Setting $x_1 = ... = x_k = 1$, we get $2^{k(k-1)}$. Multiplying this by the factor $2^{k'(k'+1)}$ from (1), we get $2^{k(k-1)+k'(k'+1)}$. It is simple to check that regardless of whether n is even or odd, the exponent $k(k-1) + k'(k'+1)$ is equal to $n(n+1)/2 - k$, as was to be shown.

IV. Completion of proof.

We can deduce the desired identity as a special case of a general formula on products of determinants. This formula appears as formula II on page 45 (chapter 3, section 9) of [9], where it is attributed to Sylvester. However, we give our own proof below.

Suppose (A^*, B^*) is a fixed partition of $\{2, 4, ..., 2k\}$ into two sets, and let $v(1), ..., v(2k)$ be *any* $2k$ row-vectors of length k. Given $A \subseteq \{1, ..., 2k\}$ with $|A| = k$, let $||A||$ denote the determinant of the k-by-k matrix

$$\begin{vmatrix} v(a_1) \\ v(a_2) \\ \vdots \\ v(a_k) \end{vmatrix},$$

where $A = \{a_1, a_2, ..., a_k\}$ with $a_1 < a_2 < ... < a_k$. Abusing terminology somewhat, we will sometimes think of A as a set of vectors $v(a_i)$, rather than as a set of integers a_i.

Theorem 2:

$$\sum_{(A,B)} ||A|| \cdot ||B|| = ||\{1, 3, ..., 2k-1\}|| \cdot ||\{2, 4, ..., 2k\}|| \quad (4)$$

where (A, B) ranges over all balanced partitions of $\{1, 2, ..., 2k\}$ with $A \cap \{2, 4, ..., 2k\} = A^*, B \cap \{2, 4, ..., 2k\} = B^*$.

Remark: This yields as a corollary the desired formula of the last section, with an extra sign-factor everywhere to take account of the fact that we are stacking row-vectors the other way.

Proof of Theorem 2: Every term on the left is linear in $v(1), ..., v(2k)$, as is the term on the right; hence it suffices to check the identity when all the $v(i)$'s are basis vectors for the k-dimensional row-space.

First, suppose that the list $v(1), ..., v(2k)$ does *not* contain each basis vector exactly twice. Then it is easy to see that every term vanishes.

Now suppose that the list $v(1), ..., v(2k)$ contains each basis vector exactly twice. There are then 2^k ways to partition $\{1, ..., 2k\}$ into two sets A, B of size k such that $||A|| \, ||B|| \neq 0$, since for each of the k basis vectors we get to choose which copy goes into A and which goes into B. However, not all of these partitions occur in the sum on the left, since we are limited to partitions (A, B) with $A \supseteq A^*, B \supseteq B^*$. Call such balanced partitions **good**.

Suppose that the basis vectors $v(1), v(3), ..., v(2k-1)$ are not all distinct; say $v(s) = v(t)$ with s, t odd, $s < t$. Then, for every good balanced partition (A, B) that makes a nonzero contribution to the left-hand side, we must have $s \in A$ and $t \in B$ or vice versa. But then $(A \triangle \{s, t\}, B \triangle \{s, t\})$

(where $\triangle$ denotes symmetric difference) is another good balanced partition. We claim that it cancels the contribution of (A, B). For, if one simply switches the row-vectors $v(s)$ and $v(t)$, one introduces $t - s - 1$ inversions, relative to the prescribed ordering of the rows in the determinant; specifically, each i with $s < i < t$ has the property that $v(i)$ is out of order relative to whichever of $v(s), v(t)$ is on the same size of the new partition. Ordering the row-vectors properly introduces a sign of $(-1)^{t-s-1} = -1$. This leads to cancellation.

Finally, suppose that $v(1), v(3), ..., v(2k-1)$ are all distinct, as are $v(2), v(4), ..., v(2k)$. We must check that the sole surviving term on the left has the same sign as the term on the right. This is clear in the case where $A^* = \{1, 3, ..., 2k-1\}$ and B^* is empty, for then the two terms are identical. We will prove the general case by showing that if one holds $v(1), ..., v(2k)$ fixed while varying (A^*, B^*), the sign of the left side of the equation is unaffected. For that purpose it suffices to consider the operation of swapping a single element from A^* to B^*. Say this element is s (with s odd). Then there is a unique $t \neq s$ (with t even) such that $v(s) = v(t)$. Let us swap s with t in the term on the left side of the equation; since $v(s) = v(t)$, the determinants are not affected. In performing the swap, we have introduced either $t - s - 1$ (if $t > s$) or $s - t - 1$ (if $s > t$) inversions, relative to the prescribed ordering of the rows. Since both quantities are even, we may re-order the rows in the determinants so that indices increase from top to bottom, without changing the sign of the product of the two determinants. We now recognize the modified term as the sole non-vanishing term associated with $(A^* \setminus \{s\}, B^* \cup \{s\})$. Since this term has the same sign as the term associated with (A^*, B^*), and since the sign is correct in the base case $(\{1, ..., 2k-1\}, \phi)$, the correctness of the sign for all partitions of $\{1, ..., 2k-1\}$ follows by induction.

This concludes the proof of Theorem 2, which in turn implies Theorem 1. □

REMARK. An identity equivalent to summing both sides of equation (4) for all 2^k sets A^* (with row $v(m)$ specialized to $(h_{m-k}, \ldots, h_{m-2}, h_{m-1})$ as needed for Theorem 1) is a special case of an identity proved combinatorially by M. Fulmek [3] using a nonintersecting lattice path argument. It is easily seen that Fulmek's proof applies equally well to prove our Theorem 2. Thus Fulmek's paper contains an implicit bijective proof of Theorem 2.

V. Probabilistic application.

One can define a probability distribution on ordered partitions of $\{1, 2, ..., 2k\}$ into two sets of size k, where the probability of the partition (A, B) is

$$\frac{\left(\prod_{1 \le i < j \le k} \frac{a_j - a_i}{j - i}\right)\left(\prod_{1 \le i < j \le k} \frac{b_j - b_i}{j - i}\right)}{2^{k^2}}.$$

Theorem 1 is equivalent to the assertion that the k random events $2 \in A, 4 \in A, ..., 2k \in A$ are jointly independent, and it is in this connection that it was first noticed. As a weakening of this assertion, we may say that the events $s \in A$ and $t \in A$ are uncorrelated with one another when s and t are both even (or both odd, by symmetry).

Theorem 3: For $1 \le m \le k$, let N_m be the random variable $|A \cap \{1, ..., m\}|$, where (A, B) is a random partition of $\{1, ..., 2k\}$ in the sense defined above. Then N_m has mean $m/2$ and standard deviation at most $\sqrt{m/2}$.

Proof: Define indicator random variables

$$I_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \in B, \end{cases}$$

so that $N_m = I_1 + I_2 + ... + I_m$. Each I_i has expected value $1/2$, by symmetry, so the expected value of N_m is $m/2$. To estimate the variance, split up the terms of N_m into $N_m^{\text{odd}} = I_1 + I_3 + ...$ and $N_m^{\text{even}} = I_2 + I_4 + ...$ The terms in each sum are independent random variables of variance $\frac{1}{4}$, so the variance of N_m^{odd} is $\frac{1}{4}\lceil m/2 \rceil$ and the variance of N_m^{even} is $\frac{1}{4}\lfloor m/2 \rfloor$. It follows from the Cauchy-Schwarz inequality that the standard deviation of N_m is at most $\sqrt{\frac{1}{4}\lceil m/2 \rceil} + \sqrt{\frac{1}{4}\lfloor m/2 \rfloor} \le \sqrt{m/2}$, as was to be shown.

The significance of the random variables N_m is that (up to an affine renormalization) they are values of the "height-function" associated with a random domino-tiling of the Aztec diamond (see [1]). Theorem 3 tells us that if one looks along the spine, the sequence of differences between heights of consecutive vertices satisfies a weak law of large numbers.

VI. Open problems.

One open problem is to find a combinatorial (preferably bijective) proof of Theorem 1. For instance, one might be able to find a bijection between the tilings compatible with (A^*, B^*) and the tilings compatible with some other partition of $\{2, 4, ..., 2k\}$.

Also, recall the variables $x_1, x_2, ..., x_m$ that made a brief appearance in section III before getting swallowed up by the notation. Leaving aside our appeal to the explicit formulas for $s_\sigma(x_1, ..., x_k, 0, ...)$ and $s_\tau(x_1, ..., x_k, 0, ...)$, we may use the linear algebra formalism of section IV to derive a Schur function identity in infinitely many variables, expressing the product $s_\sigma s_\tau$ as a sum of products of other pairs of Schur functions. It would be desirable to have a combinatorial explanation of these identities at the level of Young tableaux.

In section V, we made use of the fact that if (A, B) is chosen randomly from among the balanced ordered partitions of $\{1, 2, ..., 2k\}$, and if $s, t \in \{1, 2, ..., 2k\}$ have the same parity, then the events $s \in A$ and $t \in A$ are independent of one another. We conjecture, based on numerical evidence, that if $s, t \in \{1, 2, ..., 2k\}$ have opposite parity, then the events $s \in A$ and $t \in A$ are negatively correlated. This conjecture is made plausible by the fact that the total cardinality of A is required to be k. With the use of this conjecture, one could reduce the bound on the standard deviation in Theorem 3 by a factor of $\sqrt{2}$. However, neither Theorem 3 nor this strengthening of it comes anywhere close to giving a true estimate of the variance of N_m, which empirically is on the order of $\log k$ or perhaps even smaller.

Finally, fix $1 \leq k \leq n$. Define 0-1 random variables $X_1, X_2, ..., X_n$ such that for all $(x_1, x_2, ..., x_n) \in \{0,1\}^n$, $\text{Prob}[X_i = x_i \text{ for all } i] = 0$ unless $\sum_{i=1}^n x_i = k$, in which case

$$\text{Prob}[X_i = x_i \text{ for all } i] = \frac{\left(\prod_{1\leq i<j\leq k} \frac{a_j - a_i}{j-i}\right)\left(\prod_{1\leq i<j\leq n-k} \frac{b_j - b_i}{j-i}\right)}{2^{k(n-k)}},$$

where $\{a_1, a_2, ..., a_k\} = \{i : x_i = 1\}$ and $\{b_1, b_2, ..., b_{n-k}\} = \{i : x_i = 0\}$ ($a_1 < a_2 < ... < a_k$, $b_1 < b_2 < ... < b_{n-k}$). This is the distribution on zig-zag patterns in the kth diagonal of the Aztec diamond, induced by a domino tiling chosen uniformly at random. Are the X_i's (non-strictly) negatively pairwise correlated?

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