Continuum spanning trees: proposed by James Propp

Does the uniform spanning tree model in $\mathbf{Z}^d$, studied by Russell Lyons, Robin Pemantle, Robert Burton and others, admit a continuum limit as the grid-size goes to zero?

Let us limit the question to the case $2 \leq d \leq 4$, since that is the case in which Pemantle obtains a non-trivial measure supported on the set of spanning trees of $\mathbf{Z}^d$.

One intriguing feature of the question is that it is not initially clear what sort of geometric object a continuum tree spanning $\mathbf{R}^d$ would be. Certainly it is not a point-set in $\mathbf{R}^d$, for it would have to be nothing but the space in its entirety! On the other hand, there are reasons to believe that such a continuum object would have an interestingly high degree of symmetry (rotational symmetry for all d and perhaps conformal symmetry for $d = 2$), so the reward for a successful construction might be great: it could be an important step in the attempts of mathematical physicists to understand how lattice models, taken to the continuum limit, can manifest isotropy that is not present at any finite stage.

It is best to step back from spanning trees conceived naively as sets of vertices and edges and try to find some other description that can be taken to the limit more readily. Here are three possibilities, in which successively easier compactness arguments can be applied to give constructions for the measure we desire.

(a) Given two vertices of a tree, there is a unique path in the tree from one to the other. Conversely, knowing the function that takes each pair of vertices to the associated path tells us the tree. Hence we might model a tree spanning $\mathbf{R}^d$ as a function that takes every pair $(u, v) \in (\mathbf{R}^d)^2$ to a path joining u and v. Recent (and still unpublished) work of Itai Benjamini and Oded Schramm and of Harry Kesten shows that for $d = 2$ and $d = 3$, but not for $d = 4$, the path joining the grid-points $\mathbf{x}$ and $\mathbf{y}$ in a uniform spanning tree of $\mathbf{Z}^d$ tends to stray from $\mathbf{x}$ and $\mathbf{y}$ by no more than a distance that is linear in the separation of the two points. Using this "tightness" result and a compactness argument, one can show that a (possibly non-unique) weak limit exists for the law of this path as the grid-size goes to zero. This is the same as the continuum limit of loop-erased random walk. Note that it is possible that in the continuum limit, loop-erased random walk is self-intersecting. Can one prove anything about this? (It should be mentioned that recent work of Chad Fargason on Brownian motion is essentially equivalent to the work of Benjamini, Schramm, and Kesten.)

(b) Given three vertices of a tree, not lying on a single path, there is a unique vertex at which they "meet" (if the three vertices lie on a path, it is the middle vertex of the three; otherwise, the meeting-point is the unique vertex in the tree from which there exist edge-disjoint paths to each of the three vertices). Thus we could model a tree spanning $\mathbf{R}^d$ as a function from $(\mathbf{R}^d)^3$ to $\mathbf{R}^d$. (In the case $d = 2$ we might want the range to be $\mathbf{R}^2 \times \{+1, -1\}$, where the sign specifies the handedness of the meeting-point, relative to the ordering of the three vertices.) Benjamini and Schramm have looked at this in the case of the "pillow-case graph", consisting of two large grid-squares identified along their boundaries. This graph is asymptotically conformally equivalent to a sphere. Three vertices of the graph, along with their meeting-point, thus determine four points on the sphere, with a unique cross-ratio. Benjamini and Schramm have done empirical work that strongly suggests that, asymptotically, the probability distribution governing this cross-ratio does not depend on which three points on the pillow-case graph are chosen. This supports the conjecture that the continuum limit shows conformal invariance. It would be interesting to see if the handedness-bit is distributed in a $\frac{1}{2}, \frac{1}{2}$ way, as conformal invariance would require. For $d = 2$ and $d = 3$, the tightness result from (a) implies the tightness we need here to deduce the existence of a (not necessarily unique) sub-sequential limit-distribution; for $d = 4$, tightness fails, and the meeting-point does not have a limiting distribution (under the appropriate normalization).

(c) Given four distinct vertices in a tree, we may look at the minimal sub-tree containing them; there are four distinct topologies this sub-tree can manifest. First, there may be some (necessarily unique) "hub" vertex admitting edge-disjoint paths to all four vertices (the hub might or might not be one of those four vertices themselves). Or, if this is not the case, then there must be a unique way to partition the four vertices into two pairs such that the vertices in each pair are joined by vertex-disjoint paths (that is, out of the three *a priori* ways of dividing the four into two and two, exactly one will have this property). Therefore we may model a tree spanning $\mathbf{R}^d$ as a function from $(\mathbf{R}^d)^4$ to a four-element set, where the four elements correspond to the aforementioned topologies on the sub-tree that has the specified vertices as leaves. It is not hard to show that this information allows one to reconstruct the tree in its entirety. Also, Schramm has pointed out to me a simple random walk argument that shows that the first of the four topologies (that is, the case in which there exists some vertex admitting edge-disjoint paths to all four vertices) has probability zero in the continuum limit. In this case a 4-tuple in $\mathbf{R}^d$ would determine a triple of probabilities summing to 1. It would be interesting to check whether this distribution manifests rotational invariance (and, in the case $d = 2$, conformal invariance).

In each case, there might be a way to show, without actually giving an explicit construction, that the weak limit is unique. In any event, numerical experiments are easy to perform, using loop-erased random walk for efficient generation of spanning trees of finite graphs.