A Frosh-Friendly Completeness Axiom for the Reals

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When, and how, shall we teach our students about the completeness of the reals?

To some extent, the "how" dictates the "when", since if our approach to the completeness property is complicated, we will rightly defer it until students have attained an appropriate degree of mathematical maturity. (There is also a subtle converse force at work: for, if coming to grips with the difficulties of real analysis serve as a coming-of-age ritual for our students, we will not want to make things too simple for them; learning how to advance in the face of difficulties becomes part of the agenda of the introductory real analysis course.)

The purpose of this note is to point out a simple completeness axiom that is simple to state, is intuitively convincing to the average first-year college student, and suffices for the purposes of proving the deeper first-year calculus facts such as the intermediate value theorem.

THE CUT AXIOM: If A, B are disjoint non-empty sets whose union is $\mathbf{R}$, such that $a < b$ for all $a \in A$ and $b \in B$ (a condition we might write more compactly as $A < B$), then there exists a cutpoint $c \in \mathbf{R}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$ (a conclusion we can write more compactly as $A \leq c \leq B$).

These partitions of $\mathbf{R}$ are close in spirit to Dedekind cuts, but the sets partition $\mathbf{R}$, not $\mathbf{Q}$, and we are axiomatizing the reals, not constructing them.

The Cut Axiom is a special case of Dedekind's least upper bound property (see below for more on this), but it is easier to picture a set like A than a fully general subset of $\mathbf{R}$, and use of the Cut Axiom makes it possible to defer discussion of least upper bounds, which are hard for undergraduates to understand on a first encounter. Other completeness axioms for $\mathbf{R}$ are the convergence of bounded monotone sequences and the nested interval property, but the former requires notions of sequence convergence that are usually not introduced until quite late in the year (usually the second semester), and neither of these axioms is as simple as the Cut Axiom.

One can introduce the Cut Axiom during the first week of the class, prior to discussing functions and continuity. Indeed, if one wants to elicit class participation early in the term with a juicy question (thereby establishing norms of high student involvement), asking the question "How can you partition the real number line into two sets, one of which is completely to the left of the other?" is a good way to get the ball rolling. Once the students have wrestled with the question for a while, and haven't found any such partitions besides those of the form $A = (-\infty, c]$, $B = (c, \infty)$ and $A = (-\infty, c)$, $B = [c, \infty)$, they're glad to accept the axiom. (If students suggest taking $A = (-\infty, .999\ldots]$ and $B = [1, \infty]$, one can defer the issue of whether $.999\ldots = 1$ by pointing out that if $.999\ldots$ and 1 are different numbers, then their average differs from both of them and belongs to neither $A = (-\infty, .999\ldots]$ nor $B = [1, \infty]$.)

Speaking of $.999\ldots$: one of the first things one should use the Cut Axiom for is to show that the number line can't accomodate a real number that is bigger than .9, .99, .999, etc. but less than 1. If x were such a number, the positive real number $1 - x$ would have to be less than $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, and indeed less than $1/n$ for every positive integer n. Could such a positive real number exist?

Archimedean Property: If $\epsilon > 0$, there exists a positive integer n such than $1/n < \epsilon$.

Proof: Let B consist of all real numbers r for which there exists n with $1/n < r$, and A consist of everything else (negative numbers, zero, and positive numbers that violate the Archimedean property). Let c be the cutpoint between A and B; since $0 \in A$, $c \geq 0$. If $c = 0$, we are done. If $c > 0$, then $c/2 < c < 2c$, with $c/2 \in A$ and $2c \in B$. But since $2c \in B$, there exists n with $1/n < 2c$, implying $1/(4n) < c/2$, contradicting $c/2 \in A$.

A simple corollary of the Cut Axiom that is often more directly applicable in proofs is:

The Cut Property for Intervals: If A, B are disjoint non-empty sets whose union is a non-empty closed interval I such that $A < B$, then there exists a cutpoint $c \in I$ such that $A \leq c \leq B$.

Proof: Apply the Cut Axiom to A' and B', where A' consists of A and everything to the left of I and B' consists of B and everything to the right of I.

After continuity has been introduced, one can use the Cut Axiom to prove the Intermediate Value Theorem. Pedagogically this might be best done by first proving the IVT in the special case where f is increasing:

Weak Intermediate Value Theorem: If f is increasing and continuous on $[a, b]$ with $f(a) < 0 < f(b)$, then there exists c in (a, b) with $f(c) = 0$.

Lemma: If f is continuous on $I = [a, b]$ and positive at c with $a < c < b$, then f is positive on $(c - \delta, c + \delta)$ for some $\delta > 0$. (Likewise with "positive" replaced by "negative".)

Proof: Taking $\epsilon = f(c) > 0$ in the definition of continuity, we see that there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < f(c)$, which in turn implies $f(x) > 0$.

Proof of Weak Intermediate Value Theorem: Let $A = \{x \in I : f(x) < 0\}$ and $B = \{x \in I : f(x) \geq 0\}$, which are easily shown to satisfy the hypotheses of the Cut Property for Intervals. Therefore there exists c with $A \leq c \leq B$. If $f(c) < 0$, then by the Lemma, f is negative on $(c - \delta, c + \delta)$ for some small positive δ, contradicting the fact that every element of A is less than or equal to c. If $f(c) > 0$, we get a similar contradiction. Hence we must have $f(c) = 0$, as claimed.

With this warm-up accomplished, one can proceed to the usual version of the IVT:

Intermediate Value Theorem: If f is continuous on $I = [a, b]$ with $f(a) < 0 < f(b)$, then there exists c in (a, b) with $f(c) = 0$.

Proof: Let $A = \{x \in I : f \text{ is negative on all of } [a, x]\}$ and $B = I \setminus A$; these are non-empty sets satisfying $A < B$, so the Cut Property for Intervals implies that there exists c with $A \leq c \leq B$. As in the preceding proof, both $f(c) < 0$ and $f(c) > 0$ lead to a contradiction, so $f(c) = 0$. (Of course a proof for students would be more detailed.)

Later in the semester, in the discussion of maxima and minima, one will want to use the completeness of the reals to show that a continuous function on $[a, b]$ achieves its supremum and infimum. At this stage one needs to introduce least upper bounds and greatest lower bounds, and one can show that the Cut Axiom is equivalent to the Least Upper Bound Property. (To derive the Least Upper Bound Property from the Cut Axiom, let B be the set of upper bounds for the set S in question and let $A = \mathbf{R} \setminus B$. To go the other direction, simply take $S = A$.) Once the existence of least upper bounds and greatest lower bounds has been proved, it is easy to use the Cut Axiom to show that a function f that is continuous on $[a, b]$ with supremum M on $I = [a, b]$ achieves the value M somewhere in I: let B be the set of $x \in I$ such that the supremum of f on $[a, x]$ is M and let A be $I \setminus B$.

Also, after the Least Upper Bound Property has been proved, it's appropriate to show that the least upper bound of the numbers .9, .99, .999, ... is the number 1 (further exorcising the ghost of $.999\ldots < 1$).

Finally, in the second semester, when convergence of sequences is discussed, one can use the Least Upper Bound property to show that every bounded increasing sequence has a limit, in the usual way. (One could derive the fact about bounded increasing sequences directly from the Cut Axiom by letting B be the set of upper bounds and A be $\mathbf{R} \setminus B$, but this seems less natural than using the Least Upper Bound property.) At this point one can show that the limit of the sequence .9, .99, .999, ... is 1.

How much of this sort of reasoning a teacher presents is going to depend on how much the students can handle. A good teacher wants to be honest, but not in a way that gives honesty a bad name! One approach that will be appropriate in some cases is to include a statement of the assumptions that are being made in the course and then to flag certain assertions like the Intermediate Value Theorem with the announcement "A rigorous proof of this assertion makes use of the completeness of the reals" (but to include no proof). A teacher who takes this approach needs to say what is meant by the completeness of the reals; I propose that the Cut Axiom might be a better version for this purpose than the Least Upper Bound Property.

On the other hand, for teachers who want to give their students experience with proving theorems, deriving facts of the calculus from the Cut Axiom is a good transition from high school math to higher math inasmuch as it retains at least one cookbook-ish aspect: all these derivations proceed by identifying suitable sets A and B. This uniformity will be comforting to students who are approaching serious proof-making for the first time.

One good homework problem for a class with ambitious students is to ask them to show (with the aid of a hint or two) that the Cut Axiom is equivalent to the Intermediate Value Theorem. For, suppose there were a way to cut $\mathbf{R}$ into sets $A < B$ such that A has no greatest element and B has no least element; then the function that is -1 on A and $+1$ on B would be a continuous function that changes from negative to positive without ever taking the value 0.

Some of the students may feel dissatisfied with adopting the Cut Axiom (or any of the other completeness axioms for $\mathbf{R}$) as an unproved assumption, feeling that these axioms must be derivable from the algebra they learned in high school. One way to free the students from this intellectual cul de sac is to spend a little bit of time reviewing the ordered field axioms for the reals (probably not using this terminology, though!) and to point out that $\mathbf{Q}$ satisfies the axioms just as much as $\mathbf{R}$ does. So any valid proof of the Cut Axiom for $\mathbf{R}$ would become a proof of the Cut Axiom for $\mathbf{Q}$ if one replaced the word "real" by the word "rational" throughout. But the Cut Axiom is *false* for **Q** (let A be the rationals r with $r < 0$ or $r^2 < 2$ and B be $\mathbf{Q} - A$). Hence there can be no way to derive the Cut Axiom for **R** from the ordered field axioms.

Other students may come to find the Cut Axiom strange because it doesn't accurately model the way one cuts physical objects. When one cuts a piece of string into two pieces, it makes no sense to ask which of the two pieces acquires the cutpoint and which one doesn't! We should remind our students, and ourselves, that the continuum is a very subtle thing; we have gotten farther than the Greeks did in fathoming its secrets, but there are probably important things about it that we still don't know.

In a historical vein, it should be noted that the Cut Axiom is very similar to Axiom 3 from Tarski's axiomatization of the reals [insert reference]: If A, B are non-empty sets of reals with $A < B$ then there exists at least one c with $A \leq c \leq B$.

We should also mention that the Cut Axiom is equivalent to the proposition that **R** is connected. The relation between the completeness of **R** and the connectedness of **R** has been noted before, but I am not aware of any axiomatic treatments of **R** other than Tarski's that bring connectedness to the fore.

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