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BOUNDARY-DEPENDENT LOCAL BEHAVIOR FOR 2-D DIMER MODELS

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The shape of the boundary of a finite sub-region of a grid has a strong effect on the local entropy and local statistics (frequencies of local patterns) of a random dimer configuration. For large regions these quantities appear to be locally constant almost everywhere, but for many regions the local statistics are subject to macroscopic variation. The local frequencies associated with individual bonds can be determined in closed form for important special cases, and their large-scale non-constancy is probably related to the existence of a two-parameter family of extremal Gibbs measures for the models in question.

The dimer model on a finite sub-graph of the infinite square grid can have behavior that is dramatically different from that described by Fisher[1] and Kasteleyn[2] as the natural behavior of the system in the infinite-lattice limit (obtained by assuming periodic boundary conditions on ever-large finite lattices). This has has been observed before,[3,4,5] but no attempt has heretofore been made to give a general quantitative description of the ways in which the exact shape of the boundary can make its influence felt well into the interior of the region.

It is to be stressed out the outset that boundary conditions of the sort considered here are highly non-physical. Thus, the results described here give properties of the dimer model that probably do not correspond to any physical properties of the substances that the dimer models were originally introduced to model. However, insofar as the study of exactly solved models has taken on a second life as a chapter of pure mathematics, these results may be of interest to researchers. Moreover, the concept of an "effective field" (described below) could turn out to be useful in more realistic applications of statistical mechanics to non-homogeneous structures.

Figure 1 shows a tiling of a certain shape (an "Aztec diamond of order 3") by 1-by-2 tiles ("dominoes"), and an equivalent dimer packing of the associated dual graph. Aztec diamonds were introduced in a paper of Elkies et al.,[6] where it was shown that the Aztec diamond of order n has exactly $2^{n(n+1)/2}$ tilings by dominoes. One of the proofs given there used a method called "shuffling," which gave a bijection between tilings of the diamond of order n and bit-strings of length $n(n+1)/2$. Since it is easy to generate random strings of bits, shuffling gives us a way to generate random tilings of Aztec diamonds, or equivalently, random dimer-

Fig. 1. Domino tilings and dimer configurations.

configurations on the associated finite graphs, such that all $2^{n(n+1)/2}$ configurations are equally likely.

Figure 2 shows a random domino tiling of an Aztec diamond of order 64. One can see that there is an unexpected large-scale structure to a random tiling; in particular, the regions near the corners do not look random at all.

It can be shown that, as the size of the Aztec diamond goes to infinity:

- the boundary of the "frozen" (zero-entropy) domain converges to a circle (the "arctic circle theorem"[7]), and
- the density of horizontal/vertical dominoes inside and outside the circle converge to specific (and known) position-dependent values that depend analytically on position except on the circle itself (the "arctangent law"[8]).

Moreover, preliminary computer experiments support the conjecture that in the very middle of the diamond, the "local statistics" (a term used in the mathematical literature to describe the probability law governing the presence or absence of individual bonds and combinations thereof) are precisely the maximal-entropy statistics described in the work of Burton and Pemantle.[9]

Similar effects occur for other bipartite dimer models, such as the dimer model on a honeycomb lattice,[10,11] as well as ice-type models, in the presence of suitable boundary conditions. Figure 3 shows a random tiling of a regular hexagon of side 32 by lozenges (rhombuses of side 1 with angles of 60 and 120 degrees). This tiling was generated by means of the "coupling from the past" method of Propp and Wilson.[12] The same sort of claims that apply to random domino tilings of Aztec diamonds apply here as well, including the asymptotic circularity of the boundary between the "temperate zone" and the frozen region; however, the versions of the theorems that have been proved for lozenge tilings of hexagons are not quite as strong as those proved for domino tilings of Aztec diamonds.

Fig. 2. A random domino tiling of an Aztec diamond of order 64.

Fig. 3. A random lozenge tiling of a hexagon of side 32.

Fig. 4. Staggered weighting for the square lattice.

The behaviors we see at different locations in a large randomly-tiled Aztec diamond are believed to be approximations to the behavior of equilibrium measures for staggered dimer models, where weights are associated with dimer-locations according to the scheme shown in Figure 4. Similar staggered dimer models have appeared in the literature,[13] and it is not difficult to show that there is a phase transition when any of the weights exceeds the sum of the others. However, what is special about this particular form of staggered field is that when the weights a, b, c, d satisfy $ab = cd$, an *equilibrium* state for this model is a *Gibbs state* for the original model with $a = b = c = d = 1$. (Here I am using the terminology of Ruelle.[14] That is to say, a probability distribution on dimer configurations in the plane is an equilibrium state if it maximizes entropy minus mean energy, and it is a Gibbs state if the conditional distributions that arise when all but finitely many of the bonds are held fixed are Boltzmann distributions with respect to the energy-differences between configurations.)

Henry Cohn, Rick Kenyon, and this author have worked out a general formalism in which the local behavior for domino tilings of a large region of general shape can be viewed as jointly constituting the solution of a particular variational problem, in which the quantity being maximized is the integral of an entropy-like quantity associated with the local behavior seen in different parts of the region (for which an exact formula can be given), and where these local behaviors are constrained to satisfy certain global relationships arising from the nonexistence of defects. (For a discussion of the context in which these global relationships arise, see the discussion of "height functions" that were introduced in the physics literature[15,16] and independently in the mathematics literature.[17]) The solution to this variational problem can be embodied in a position-dependent "effective field" that, at each point in the region, summarizes the effects of the boundary constraints by a 4-tuple (a, b, c, d) of effective bond-weights satisfying $ab = cd$.[18]

One feature of this perspective is that the non-analytic nature of the local behavior on the critical circle in the Aztec diamond (described earlier) is seen to result from an *analytic* effective field; non-analyticity of the local behavior is the result of the fact that, on the critical circle, one of the four weights equals the sum of the other three, so that the aforementioned phase transition occurs.

Acknowledgements

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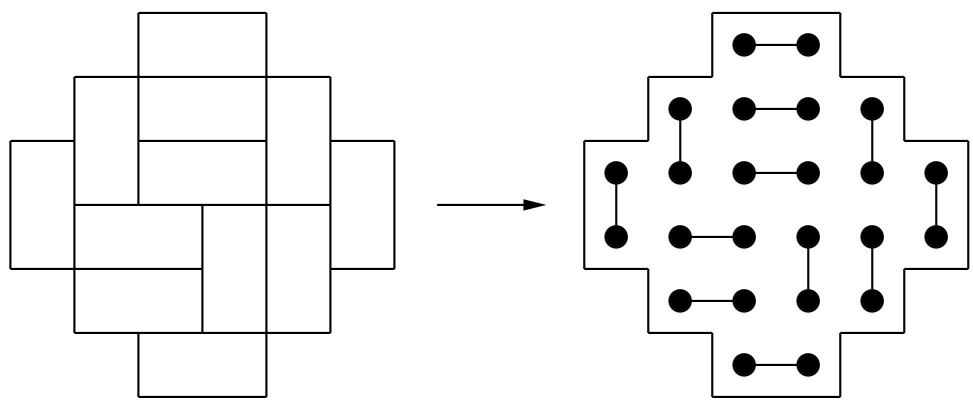

Figure 1

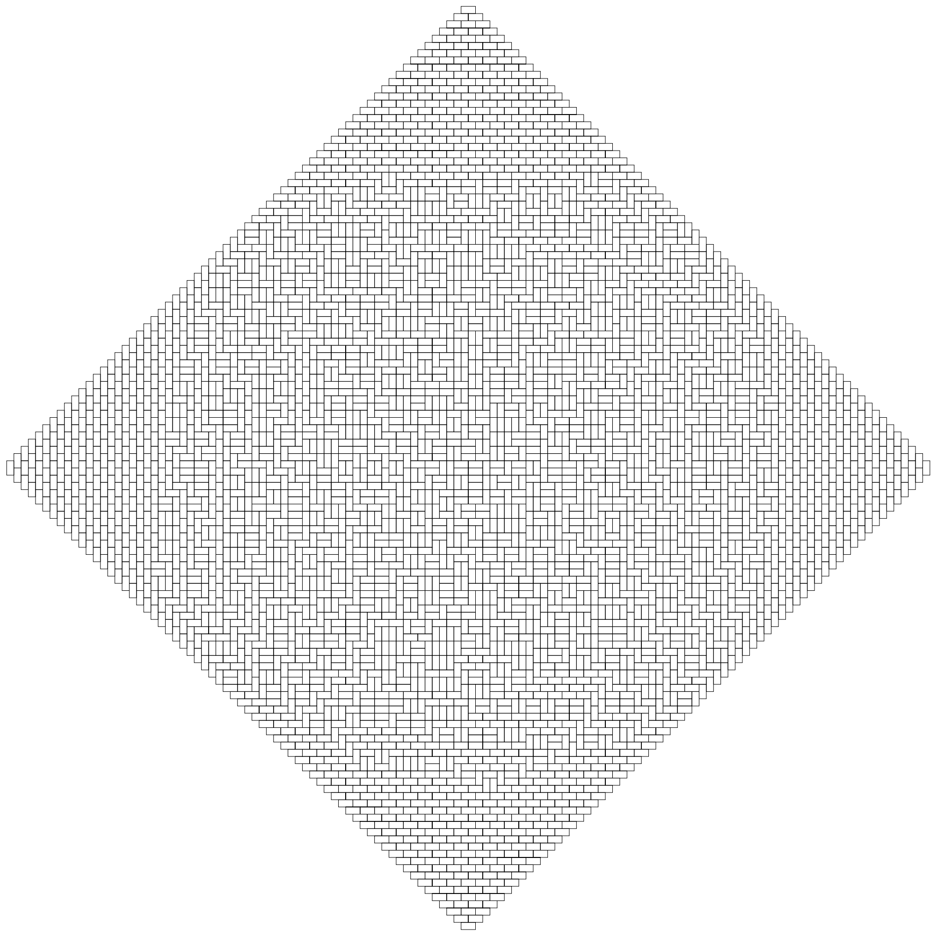

Figure 2

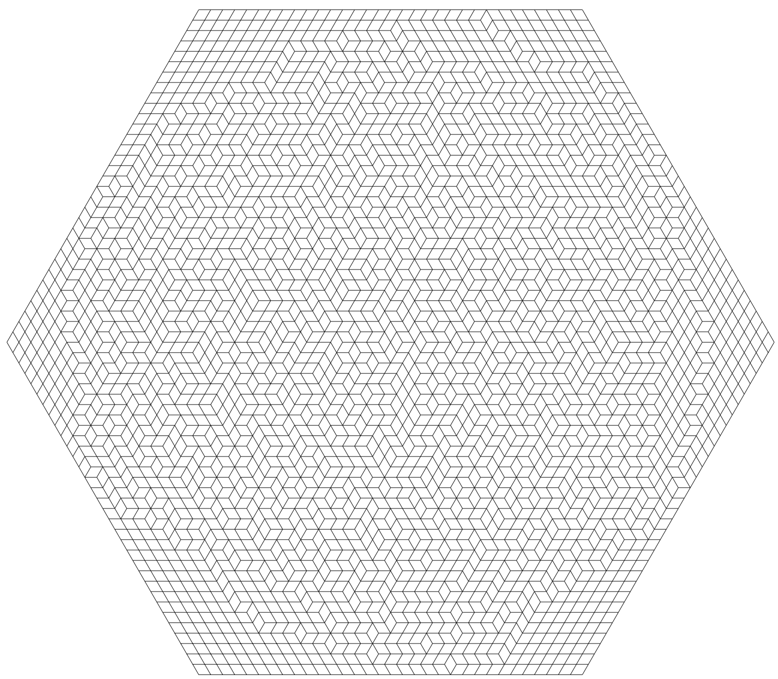

Figure 3

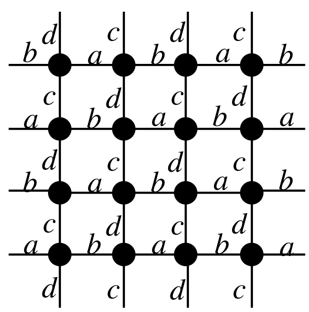

Figure 4