The combinatorics of frieze patterns and Markoff numbers

(math.wisc.edu/∼propp/fpsac06-slides.pdf)

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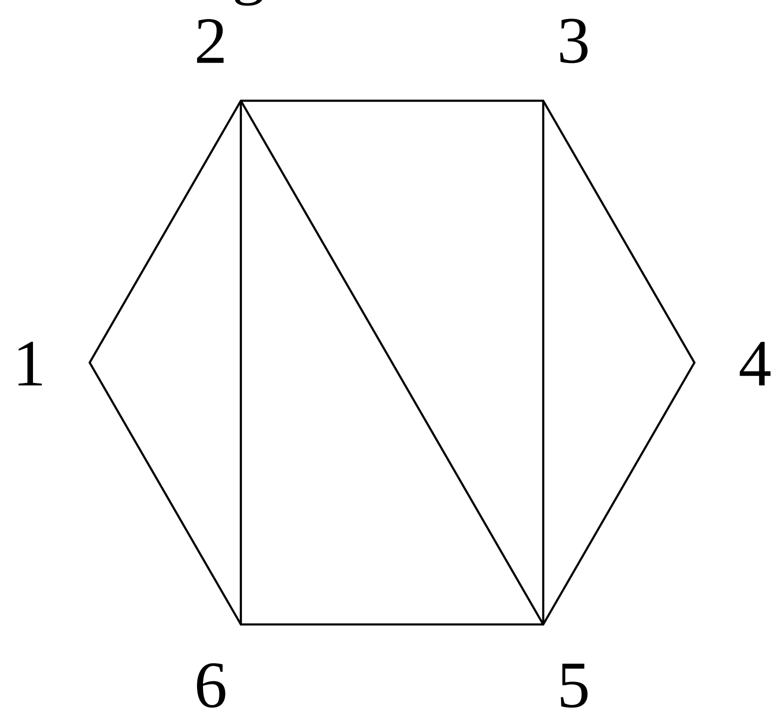
This talk describes joint work with Dylan Thurston and with (former or current) Boston-area undergraduates Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, and Rui Viana, under the auspices of REACH (Research Experiences in Algebraic Combinatorics at Harvard). For details of the proofs, see math.CO/0511633.

I. Triangulations and frieze patterns

To every triangulation T of an n-gon with vertices cyclically labelled 1 through n, Conway and Coxeter associate an $(n-1)$-rowed periodic array of numbers called a **frieze pattern** determined by the numbers $a_1, a_2, \ldots, a_n$, where a_k is the number of triangles in T incident with vertex k.

(See J. H. Conway and H. S. M. Coxeter, "Triangulated Polygons and Frieze Patterns," *Math. Gaz.* **57** (1973), 87–94 and J. H. Conway and R. K. Guy, in *The Book of Numbers*, New York : Springer-Verlag (1996), 75–76 and 96–97.)

E.g., the triangulation



of the 6-gon determines the 5-row frieze pattern

```
... 1 1 1 1 1 1 1 1 1   ...
...  1 3 2 1 3 2 1 3 2 ...
... 1 2 5 1 2 5 1 2 5   ...
...  1 3 2 1 3 2 1 3 2 ...
... 1 1 1 1 1 1 1 1 1   ...
```

Rules for constructing frieze patterns:

1. The top row is

$$\ldots, 1, 1, 1, \ldots$$

2. The second row (offset from the first) is

$$\ldots, a_1, a_2, \ldots, a_n, a_1, \ldots$$

(with period n).

3. Each succeeding row (offset from the one before) is determined by the recurrence

```
  A
 B C  :   D = (BC - 1) / A
  D
```

Facts:

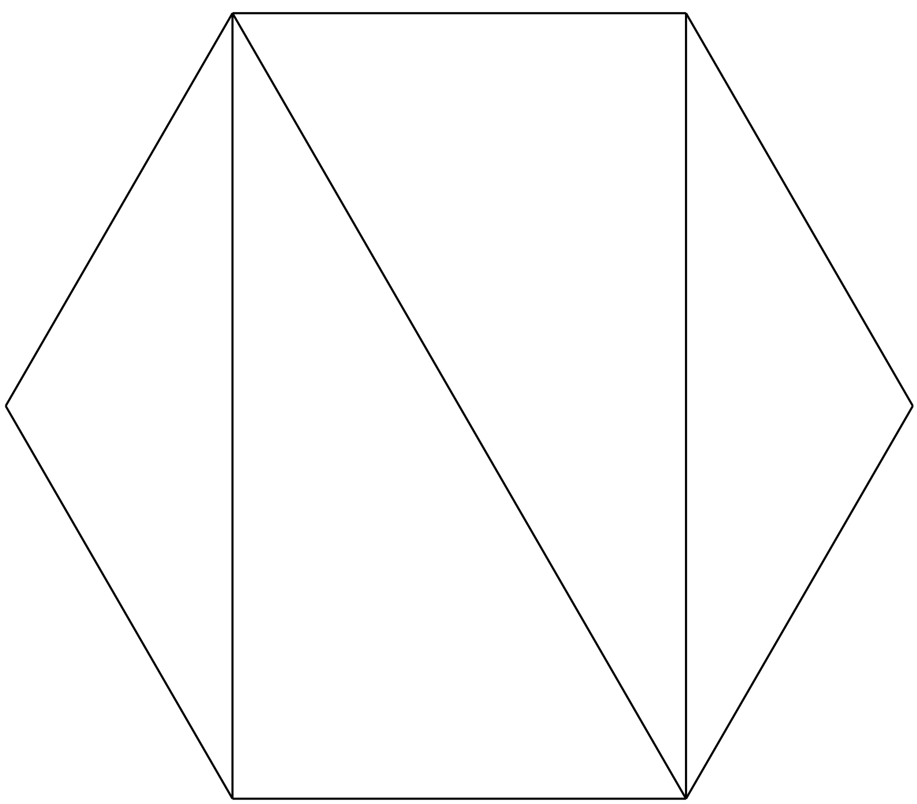
- Every entry in rows 1 through $n-1$ is non-zero (so that the recurrence

```
D = (BC-1)/A
```

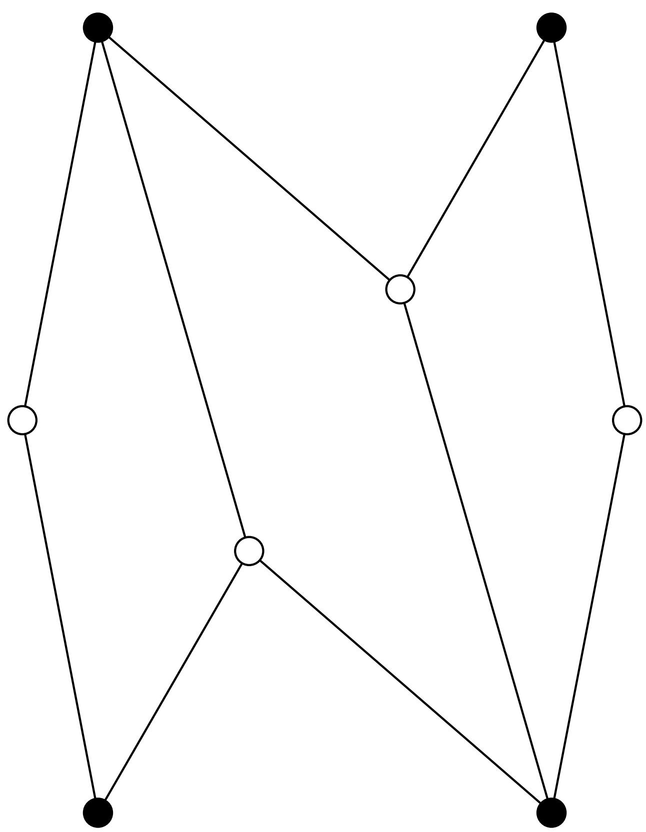
  never involves division by 0).
- Each of the entries in the array is a positive integer.
- For $1 \leq m \leq n-1$, the $n-m$th row is the same as the mth row, shifted. (That is, the array as a whole is invariant under a glide reflection.)

Question: What do these positive integers count? (And why does the array possess this symmetry?)

E.g., in the following picture, what are there 5 of?

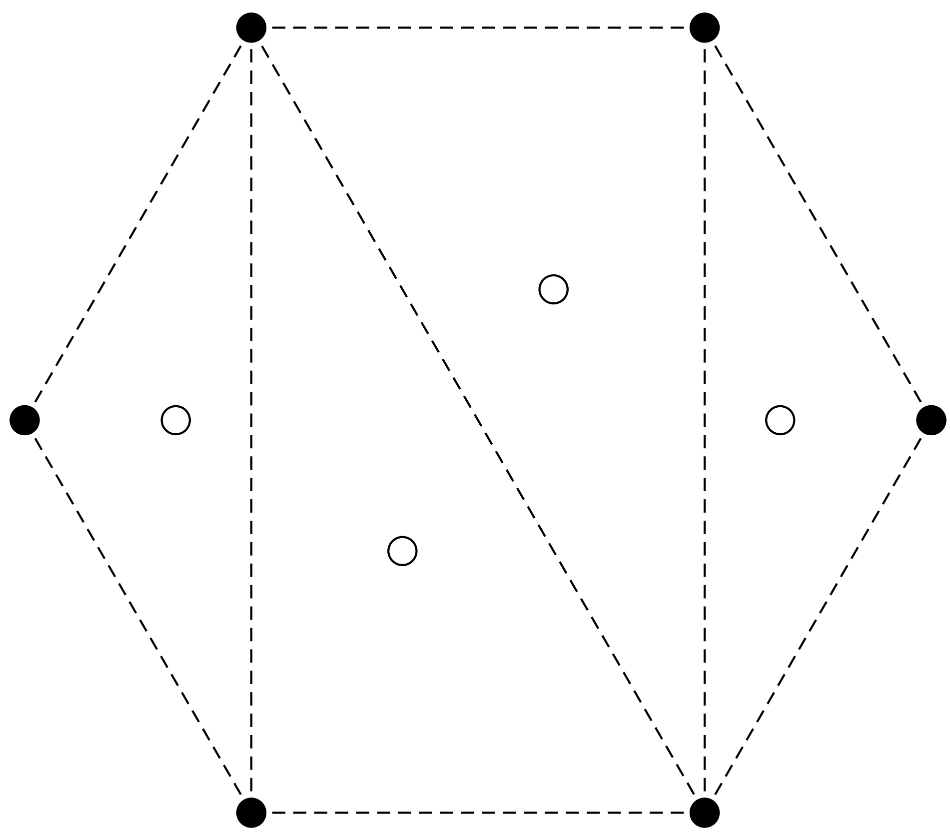


Answer: Perfect matchings of the graph

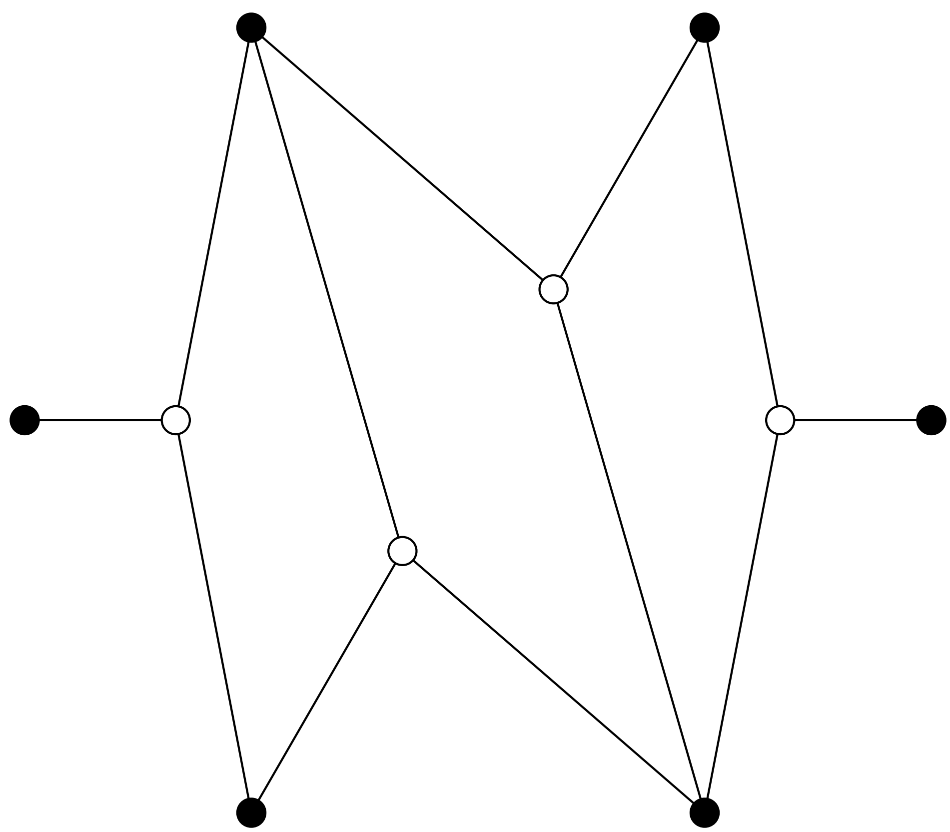


General construction:

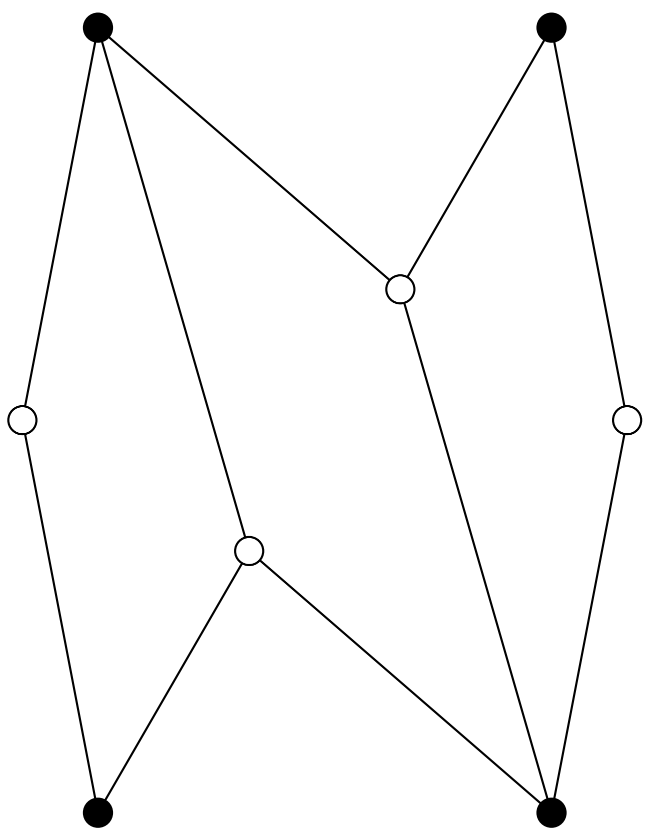
Put a black vertex at each of the n vertices of the n-gon.
Put a white vertex in the interior of each of the $n-2$ triangles in the triangulation T.



For each of the $n - 2$ triangles, connect the black vertices of the triangle to the white vertex inside the triangle. This gives a connected planar bipartite graph with n black vertices and $n - 2$ white vertices.



If we remove 2 of the black vertices (say vertices i and j), we get a graph with equally many black and white vertices. Let $C_{i,j}$ be the number of perfect matchings of this graph.



Theorem (Gabriel Carroll and Gregory Price): The Conway-Coxeter frieze pattern is just

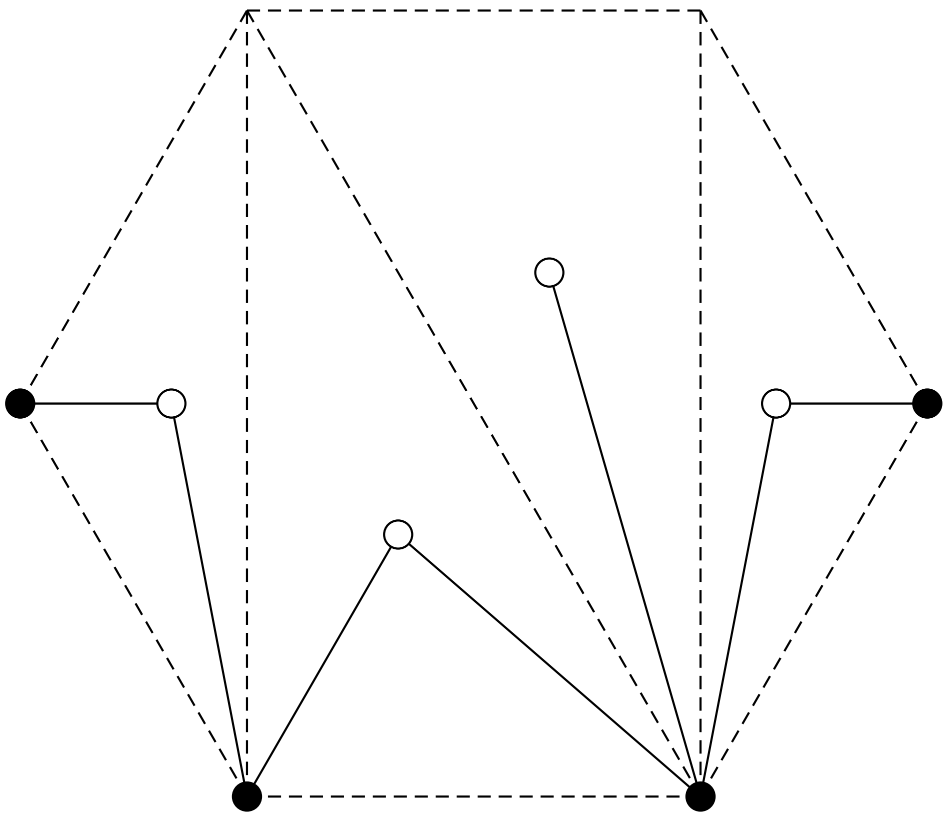
$$\begin{array}{ccccccccccc}
\ldots & C_{1,2} & & C_{2,3} & & C_{3,4} & & C_{4,5} & \ldots \\
\ldots & & C_{1,3} & & C_{2,4} & & C_{3,5} & & \ldots \\
\ldots & C_{n,3} & & C_{1,4} & & C_{2,5} & & C_{3,6} & \ldots \\
\ldots & & C_{n,4} & & C_{1,5} & & C_{2,6} & & \ldots \\
& \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

(interpret all subscripts mod n).

Note: This claim explains the glide-reflection symmetry.

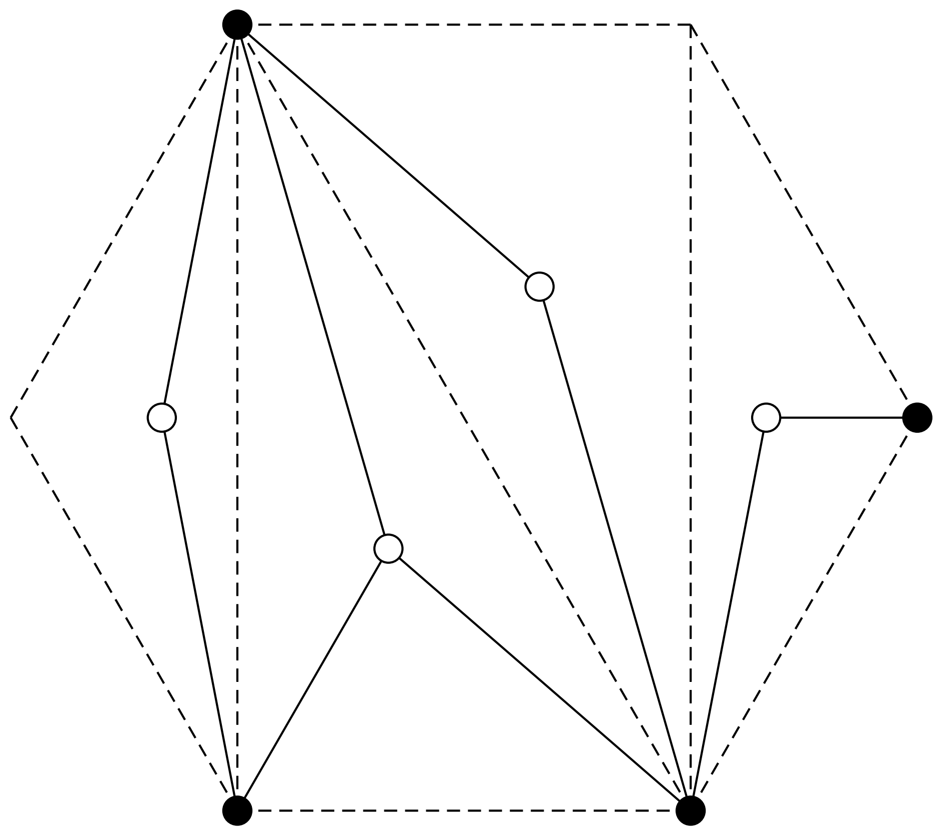
Proof of theorem:

1. $C_{i,i+1} = 1$.



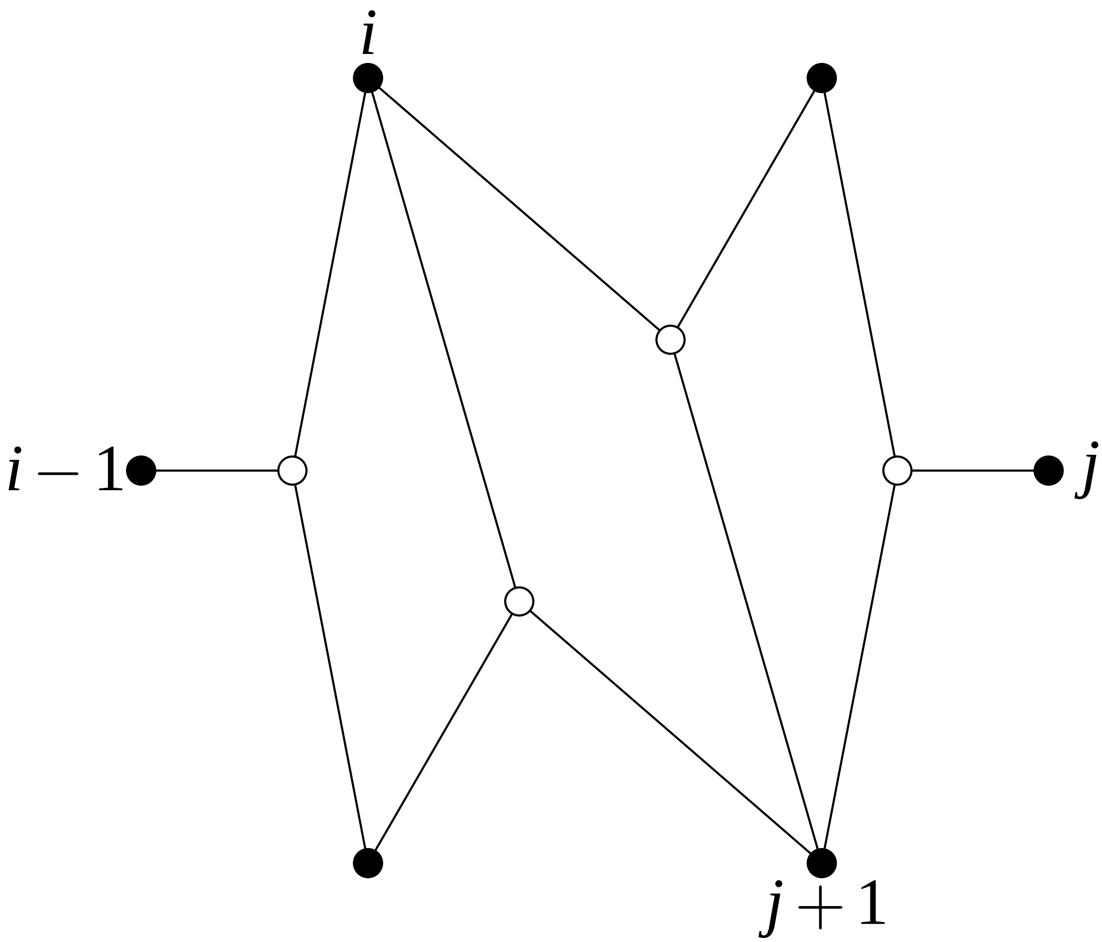
(proof of theorem, continued)

2. $C_{i-1,i+1} = a_i$.



(proof of theorem, continued)

3. $C_{i,j}C_{i-1,j+1} = C_{i-1,j}C_{i,j+1} - 1.$



Move the 1 to the left-hand side, and write the equation in the form

$$C_{i,j}C_{i-1,j+1} + C_{i-1,i}C_{j,j+1} = C_{i-1,j}C_{i,j+1}$$

(proof of theorem, concluded)

This is a consequence of a lemma due to Eric Kuo (see Theorem 2.5 in "Applications of graphical condensation for enumerating matchings and tilings," `math.CO/0304090`):

If a bipartite planar graph G has 2 more black vertices than white vertices, and black vertices a, b, c, d lie in cyclic order on some face of G, then

$$M(a,c)M(b,d) =$$
$$M(a,b)M(c,d) + M(a,d)M(b,c),$$

where $M(x,y)$ denotes the number of perfect matchings of the graph obtained from G by deleting vertices x and y and all incident edges.

A version of this construction that includes edge-weights gives the cluster algebras of type A introduced by Sergey Fomin and Andrei Zelevinsky. (See section 3.5 of Fomin and Zelevinsky, "Y-systems and generalized associahedra", `hep-th/0111053`.)

In this broadened context, the entries of frieze patterns are rational functions rather than numbers. Fomin and Zelevinsky proved that these rational functions are Laurent polynomials.

The matchings model can be used to show that the coefficients in these Laurent polynomials are all positive (as was conjectured by Fomin and Zelevinsky).

II. Markoff numbers

A **Markoff triple** is a triple (x,y,z) of positive integers satisfying $x^2+y^2+z^2 = 3xyz$; e.g., the triple (2,5,29).

A **Markoff number** is a positive integer that occurs in at least one such triple.

Writing the Markoff equation as

(*) $\qquad z^2-(3xy)z+(x^2+y^2)=0,$

a quadratic equation in z, we see that if (x,y,z) is a Markoff triple, then so is (x,y,z'), where $z' = 3xy - z = (x^2+y^2)/z$, the other root of (*).

(z' is positive because $z' = (x^2+y^2)/z$, and is an integer because $z' = 3xy - z$.)

Likewise for x and y.

Claim: Every Markoff triple (x,y,z) can be obtained from the Markoff triple $(1,1,1)$ by a sequence of such exchange operations. E.g., $(1,1,1) \to (2,1,1) \to (2,5,1)$ $\to (2,5,29)$.

Proof idea: Use high-school algebra and some Olympiad-level cleverness to show that if (x,y,z) is a Markoff triple with $x \geq y \geq z$, and we take $x' = (y^2+z^2)/x$, then $x' < x$ unless $x = y = z = 1$. See A. Baragar, "Integral solutions of the Markoff-Hurwitz equations," (*Journal of Number Theory* **49** (1994), 27–44).

So in fact, each Markoff triple can be obtained from (1,1,1) by a sequence of moves that leaves two numbers alone and increases the third.

Create a graph whose vertices are the Markoff triples and whose edges correspond to the exchange operations

$$(x,y,z) \longrightarrow (x',y,z),$$

$$(x,y,z) \longrightarrow (x,y',z),$$

$$(x,y,z) \longrightarrow (x,y,z')$$

where

$$x' = \frac{y^2+z^2}{x},$$

$$y' = \frac{x^2+z^2}{y},$$

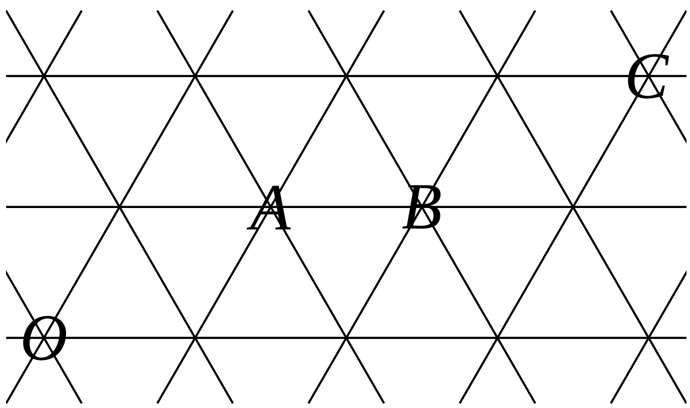
$$z' = \frac{x^2+y^2}{z}.$$

This 3-regular graph is connected (see the preceding claim), and it is not hard to show that it is acylic. Hence the graph is the 3-regular infinite tree.

Markoff numbers are associated with pairs of mutually visible lattice points in the triangular lattice. This association is bijective (up to lattice symmetry).

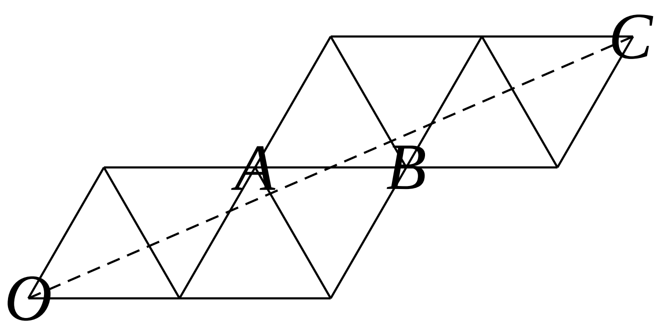
Equivalently, we can associate Markoff numbers (up to symmetries of the triangular lattice L) with primitive vectors in L, where a non-zero vector $\mathbf{u}$ is called **primitive** if it cannot be written as $k\mathbf{v}$ for $k > 1$ and $v \in L$.

For example, the Markoff triple $2, 5, 29$ corresponds to the three primitive vectors $\mathbf{u} = \vec{OA}$, $\mathbf{v} = \vec{OB}$, and $\mathbf{w} = \vec{OC}$, with O, A, B, and C forming a fundamental parallelogram for the triangular lattice, as shown below.



The Markoff number 1 corresponds to the primitive vector $\vec{AB}$.

To find the Markoff number associated with a primitive vector $\vec{OX}$, take the union R of all the triangles that segment OX passes through. The underlying lattice provides a triangulation of R. E.g., for the vector $\mathbf{u} = \vec{OC}$ from the previous page, the triangulation is



Turn this into a planar bipartite graph as in Part I, let $G(\mathbf{u})$ be the graph that results from deleting vertices O and C, and let $M(\mathbf{u})$ be the number of perfect matchings of $G(\mathbf{u})$. (If $\mathbf{u}$ is a shortest vector in the lattice, put $M(\mathbf{u}) = 1$.)

Theorem (Gabriel Carroll, Andy Itsara, Ian Le, Gregg Musiker, Gregory Price, and Rui Viana): If **u**, **v**, and **w** are primitive vectors in the triangular lattice L with $\pm\mathbf{u}\pm\mathbf{v}\pm\mathbf{w}=\mathbf{0}$ for a suitable choice of signs, such that any two of **u**, **v**, and **w** form a basis for L, then

$$(M(\mathbf{u}),M(\mathbf{v}),M(\mathbf{w}))$$

is a Markoff triple. Every Markoff triple arises in this fashion.

In particular, if **u** is a primitive vector, then $M(\mathbf{u})$ is a Markoff number, and every Markoff number arises in this fashion.

Proof: The base case, with

$$(M(\mathbf{e}_1), M(\mathbf{e}_2), M(\mathbf{e}_3)) = (1,1,1),$$

is clear.

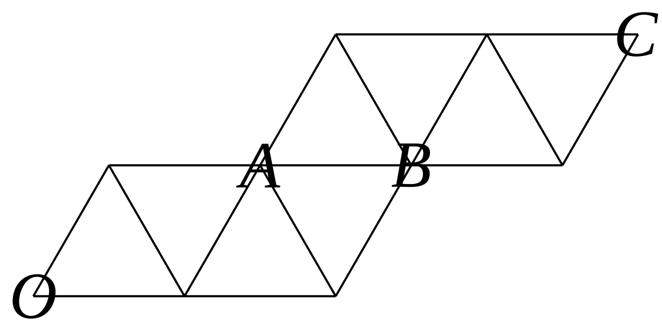
The only non-trivial part of the proof is the verification that

$$M(\mathbf{u}+\mathbf{v}) = (M(\mathbf{u})^2 + M(\mathbf{v})^2)/M(\mathbf{u}-\mathbf{v}).$$

(proof of theorem, concluded)

E.g., in the picture below, we need to verify that

$$M(\vec{OC})M(\vec{AB}) = M(\vec{OA})^2 + M(\vec{OB})^2.$$



But if we rewrite the desired equation as

$$M(\vec{OC})M(\vec{AB}) = M(\vec{OA})M(\vec{BC}) + M(\vec{OB})M(\vec{AC})$$

we see that this is just Kuo's lemma!

Remarks: Some of the work done by the REACH students used a square lattice picture; this way of interpreting the Markoff numbers combinatorially was actually discovered first, in 2001–2002 (Itsara, Le, Musiker, and Viana).

Also, the original combinatorial model for the Conway-Coxeter numbers (found by Price) involved paths, not perfect matchings. Carroll turned this into a perfect matchings model, which made it possible to arrive at the matchings model of Itsara, Le, Musiker, and Viana via a different route.

See `www.math.wisc.edu/~propp/reach/newback.jpg`.

III. Other directions for exploration

Neil Herriot (another member of REACH) showed that if we replace the triangular lattice used above by the tiling of the plane by isosceles right triangles (generated from one such triangle by repeated reflection in the sides), parallelograms of mutually visible points in the square lattice correspond to triples (x, y, z) of positive integers satisfying either

$$x^2 + y^2 + 2z^2 = 4xyz$$

or

$$x^2 + 2y^2 + 2z^2 = 4xyz.$$

So, is there some more general combinatorial approach to ternary cubic equations of similar shape?

Gerhard Rosenberger ("Uber die diophantische Gleichung $ax^2+by^2+cz^2=dxyz$," *J. Reine Angew. Math.* **305** (1979), 122–125) showed that there are exactly three ternary cubic equations of the shape $ax^2+by^2+cz^2=(a+b+c)xyz$ for which all the positive integer solutions can be derived from the solution $(x,y,z)=(1,1,1)$ by means of the obvious exchange operations $(x,y,z)\to(x',y,z)$, $(x,y,z)\to(x,y',z)$, and $(x,y,z)\to(x,y,z')$, namely:

$$x^2+y^2+z^2=3xyz,$$

$$x^2+y^2+2z^2=4xyz,$$

and

$$x^2+2y^2+3z^2=6xyz.$$

The third Diophantine equation “ought” to be associated with some combinatorial model involving the reflection-tiling of the plane by 30-60-90 triangles, but the most obvious approach (based on analogy with the 60-60-60 and 45-45-90 cases) does not work.

What about the equation $w^2+x^2+y^2+z^2=4wxyz$? (Such equations are called Markoff-Hurwitz equations.)

The Laurent phenomenon applies here too: The four exchange operations convert an initial formal solution (w,x,y,z) into a quadruple of Laurent polynomials. (This is a special case of Theorem 1.10 in Fomin and Zelevinsky's paper "The Laurent phenomenon," `math.CO/0104241`.)

The numerators of these Laurent polynomials ought to be weight-enumerators for some combinatorial model, but we have no idea what this model looks like. We can't even prove that the coefficients are positive, although they appear to be.

A variant of the notion of frieze patterns is gotten by replacing the frieze-pattern relation

```
   A
B     C   :   AD + 1 = BC
   D
```

by the relation

```
   A
B  E  C   :   AD + E = BC
   D
```

E.g.:

```
    1 1 1 1 1 1 1 1 1 1 1 1
 . . . 1 2 6 4 1 1 3 4 2 2 3 2   . . .
        4 2 2 3 2 1 2 6 4 1 1 3
          1 1 1 1 1 1 1 1 1 1 1 1
```

Conway and Hickerson have both proved that arrays of this kind have the same sort of glide-reflection symmetry as frieze patterns. Specifically, in any table of this kind with $n - 2$ rows, with top and bottom rows consisting entirely of 1's, each row has period $2n$.

All of the good algebraic properties that are satisfied by frieze patterns seem to hold for this variant as well. However, some of these properties have not been proved rigorously, and no supporting combinatorial model analogous to the matchings model of Carroll and Price is known.