A not-quite-bijective enumeration of domino tilings of Aztec diamonds

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Slides for this talk are on-line at
http://jamespropp.org/jmm12.pdf

Clever counting **or** beautiful bijection?

Answer:

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Answer: “Yes.”

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Answer: "Beautiful bijection."

Clever counting or **beautiful** bijection?

Answer: “Beautiful bijection.”
When I’m reading someone else’s work, I want it to be beautiful (with at least one “aha!” moment); “clever” can mean “tricky”!

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Answer: "Clever counting."

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Answer: “Clever counting.”
(The contrarian in me wants to go with whichever position is in the minority; the “clever counters” are seriously out-numbered today, but I didn’t know that this would happen when I submitted my abstract!)

What is a proof for?

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For individuals: conviction
For mathematical communities: cohesion
For propositions: reproduction

Slogan of this talk

A proof is a conjecture's way of generating more conjectures.

TOADs

The Aztec Diamond of order n is the union of all the lattice squares contained in the region $\{(x, y) \in \mathbf{R}^2 : |x| + |y| \leq n + 1\}$; it has area $2n(n + 1)$.

A TOAD (Tiling Of Aztec Diamond) of order n is a way of covering the Aztec diamond of order n with $n(n + 1)$ 1-by-2 and 2-by-1 rectangles ("dominos") with no gaps or overlaps (except along boundaries).

See Dan Romik's picture of a TOAD of order 5.

Counting TOADs

Let $T(n)$ be the number of TOADs of order n.

$T(0) = 1$
$T(1) = 2$
$T(2) = 8$
$T(3) = 64$
$T(4) = 1024$

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It would appear that $T(n) = 2^{n(n+1)/2}$.

Proving the conjecture

Noam Elkies gave the first proof, but it wasn't bijective.

Greg Kuperberg and I found a way to partition the set of TOADs of order $n-1$ into disjoint blocks

$$B_1, \ldots, B_M,$$

and to partition the set of TOADs of order n into equally many disjoint blocks

$$B'_1, \ldots, B'_M,$$

such that for all i,

$$|B'_i| = 2^n |B_i|.$$

Summing over i, this implies that $T(n) = 2^n T(n-1)$.

A node for every TOAD

Say a TOAD of order $n-1$ and a TOAD of order n are **compatible** if for some i they belong to B_i and B'_i, respectively.

It's helpful to imagine a directed graph structure in which the nth level ($n \geq 0$) consists of all TOADs of order n, such that there is an edge from a node at level $n-1$ to a node at level n iff the corresponding TOADs are compatible.

The equality $T(n) = 2^n T(n-1)$ (valid for every $n > 0$) implies by induction that $T(n) = 2^{n+(n-1)+\ldots+1} T(0) = 2^{n(n+1)/2}$.

Is the proof bijective?

This doesn't quite give a bijection between TOADs of order n and bit-strings of length $n(n+1)/2$.

You can modify the construction to give one, but it's a bit unnatural (you have to break a little bit of the symmetry of the problem).

Maybe not, but it's constructive!

What's important is that you can use a random walk up the graph to generate a **random TOAD** (that is, to sample from the uniform distribution on the $2^{n(n+1)/2}$ TOADs of order n).

At each step, you just choose a random upward neighbor of the current node in the graph of all TOADs.

This procedure is called **domino-shuffling**, as described in Alternating sign matrices and domino tilings by Elkies et al.

See Hal Canary's implementation.

Why generate random TOADs?

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Because random objects are generic!

What's good about generic objects?

#1. If you have a conjecture about a large finite set (or an infinite collection of finite sets), it's a good idea to test it on a generic element of the set.

What's good about generic objects?

#2. Generic objects often have surprising properties that lead one to formulate new conjectures.

Such as?

The "arctic circle phenomenon": for Aztec diamonds (and many other regions), most tilings show more order near the boundary than in the interior.

In the case of the Aztec diamond, one can draw a boundary between the orderly and disorderly domains within the tiling, and there is a sense in which it is asymptotically circular (the "arctic circle theorem").

For an example of a large randomly-tiled Aztec diamond, see Cris Moore's picture of a random TOAD of order 256.

And...

Domino shuffling has been extended by Borodin and Gorin to tilings of hexagons by rhombuses made by gluing two equilateral triangles along an edge; see Shuffling algorithm for boxed plane partitions.

For an example of a random rhombus tiling, see Ken McLaughlin's picture of a random rhombus tiling of a large hexagon.

Conclusion

The quest for bijective proofs of propositions leads to algorithms for sampling from the uniform distribution on interesting combinatorial sets; this lets us see what a typical element looks like, which leads to new propositions for us to prove.

If mathematical conjectures could vote, and if they voted in their Darwinian self-interest, I believe that they would prefer to be proved by means of **beautiful bijections**.