A Dozen Hat Problems

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Hat problems are all the rage these days, proliferating on various web sites and generating a great deal of conversation—and research—among mathematicians and students. But they have been around for quite a while in different forms. For example, a variant of Problem 3 shows up in *Puzzle-Math* by George Gamow and Marvin Stern (The Viking Press, 1958; pp. 77–78) as a question about three travelers with dirty faces. (This was EB's introduction to the field.) Martin Gardner also wrote about them in *Aha! Insight* (W.H.Freeman & Co., 1978).

In all these problems hats of specified colors are placed on players' heads. Each can see the colors of some or all of the other player's hats, but not his own, and the goal of the problem is to provide a strategy players' can agree on before the play of the game that will allow some maximal number of players to correctly guess the colors of their hats. Incorrect guesses might be ignored, or might lead to severe punishments! Players are not allowed to communicate with each other during play of the game and may only speak a single word—a guess as to the color of one's own hat—if a game permits any speaking at all! Some of these problems are psychological or philosophical and many are deeply mathematical. At the very least, they are fun to think about. Enjoy!

1. Warm-up: Two Players

Albert and Bilbert are about to play a game in which hats—either rouge or puce—will be placed on their heads. Each will be able to see the color of his partner's hat, but not the color of his own. At the blow a whistle Albert and Bilbert will simultaneously make a guess as to the color of his own hat. Incorrect guesses will not be punished. A win consists of at least one correct guess.

What strategy can Albert and Bilbert agree upon to secure a win?

2. One-hundred Players and One-hundred Colors

Following the same rules as question 1, one-hundred players will play the same game with hats that can be any of 100 different colors. (Not all colors need be used.) Each player can see the colors of her 99 colleagues. If a win consists of at least one correct guess and no penalty for incorrect guesses, devise a strategy that guarantees a win.

3. Waiting it Out

Angelina, Bettina, and Charlina sit in a circle, blindfolded. Hats—either rouge or puce—are placed on their heads. When the blindfolds are removed each lass is asked to raise a hand if she sees a rouge hat on one or both of her friends. Based solely on the information provided by these gestures—or lack thereof—is it possible for at least one woman to correctly announce the color of her own hat (if players agree to keep silent unless they are certain of their answer)? Is there ever a scenario in which this cannot be done?

Comment: It is fun to reenact this experiment by holding playing cards to one's forehead, raising a hand upon sight of a red card.

4. How Long to Wait?

One hundred blindfolded pixies sitting in a circle have hats—either rouge or puce—placed on their heads and are told that there is at least one puce hat among the group. When their blindfolds are removed, each pixie sees the hats of her 99 colleagues, but has no hint as to the color of her own hat.

A clock counts off the minutes. At each minute mark any pixie who thinks she knows the color of her own hat can speak up and make a guess. If after 100 minutes no pixie speaks, all will be eliminated. If, before then, some pixie speaks up and is incorrect, all players will be eliminated. Only if some pixies speak, all correctly, at some minute up to the 100 minute mark will the group survive. Is there a strategy the pixies can agree upon before play of this game to ensure their survival?

5. Dark Consequences

This time 100 blindfolded gnomes stand in a line, back-to-front. Each gnome's hat is either rouge or puce, and there is at least one puce hat among the gnomes (and the gnomes know this). Upon the removal of the blindfolds each gnome can see the hats of the gnomes in front of him, but not behind him. (Thus the gnome at the back of the line can see 99 hats, the gnome at the front, none.)

Starting at the back of the line, each gnome, in turn, will be asked to say either "My hat is puce" or "pass." The game will end—positively—as soon as some gnome makes the former statement and is correct! If a gnome incorrectly claims his hat is puce, all will be eliminated. If all gnomes choose to say "pass," then all shall be eliminated.

Devise a strategy that guarantees the gnomes' survival.

6. Quite Gory

Ten gnomes stand back-to-front in a line each wearing either a rouge or puce hat. There need not be a puce hat. Starting at the back of the line with the gnome who can see nine hats each gnome will make a guess as to the color of his own hat. Gnomes who guess correctly will be freed. Those that guess incorrectly will be immediately executed. (No passes in this game!) Gnomes will hear the guesses made behind them and the consequent sighs of relief or screams of horror.

Knowing that they are about to play this game, what strategy could the gnomes agree upon to ensure the survival of a maximal number of gnomes? How many gnomes?

7. Gory and Confusing

Same as question 6 but this time there are 100 different possible colors of hats!

8. Return to the Option to Pass

[This puzzle, and its solution, is due to 9th-grader Alex Smith of St. Mark's School, MA.]

Ten gnomes again stand in a line, back-to-front, each with a hat—rouge or puce—placed on his head. The color of each gnome's hat is determined by the toss of a coin (and the gnomes know this). Each gnome can see the colors of the hats in front of him.

Starting at the back of the line, each gnome is asked, in turn, to either make a guess as to the color of his hat or say PASS. If any single gnome makes an incorrect guess, all will be immediately executed. If all gnomes say pass, all will be executed. The gnomes will only survive if at least one guess is made and all guesses offered are correct.

What strategy could the gnomes agree upon to ensure them a maximal probability of survival? (Assume that each gnome can hear the guess made behind him.)

9. Three Gnomes, GNO Information

Three gnomes are about to play yet another perilous game of life and death. They will sit in a circle and an evil villain

will again place on each gnome's scalp a colored hat. The color of each hat—rouge or puce—will be determined by a coin toss. As before, each gnome will see the colors of his two colleagues' hats, but not the color of his own hat.

At the signal of the villain, the three gnomes will simultaneously each guess the color of his own hat or say pass. If at least one gnome makes a correct guess and no-one makes an incorrect guess, then all three gnomes will live. If there is at least one incorrect guess amongst the group, or if all three gnomes say pass, they will all die. There will be absolutely no communication of any form between gnomes during play of this game.

Devise a scheme that assures the gnomes 75% chance of survival as a group.

10. $2^n - 1$ Gnomes, GNO Information

Seven gnomes are about to about to play the same game as the gnomes in question 9, and then 15 gnomes, and then 31 gnomes. Devise a general scheme that gives a group of $2^n - 1$ gnomes (for n some integer) maximal chance of survival.

11. $2^n - 1$ Gnomes and a Smidgeon of Shared Information

[This variation is again due to Alex Smith.]

Repeat questions 9 and 10, but this time the gnomes can make their guesses one at a time. (Assume that they can hear each other's guesses.) Obtain much better odds of survival.

12. Absolutely no Leeway

One-hundred pixies are placed in a room and hats—either rouge or puce—are placed on their heads, with color determined by a toss of a coin. Each pixie can see all her colleagues' hats, but not her own. At the sound of a bell *each and every* pixie must make a guess as to the color of her hat—no passes—and they must all do so simultaneously. Each pixie that makes a correct guess survives, all those that make an incorrect guess die.

Devise a strategy that the pixies can agree upon before play of the game to assure the survival of a maximal number of their group. What is that maximal number?

Answers:

1. Albert guesses the hat color he sees on Bilbert and Bilbert opposite the hat color he sees on Albert.

2. Label the hat colors 0 to 99 and number the players 0 to 99. Have each player sum the colors she sees, working mod 100, and guess the color number that brings this sum up to the number she has been assigned as a player (again working mod 100). If the sum of *all* hat colors, mod 100, is s, then player s is sure to make a correct guess.

Taking it further: Prove that if N people play this game with hats of C different colors, then this strategy is sure to produce at least $\lfloor N/C \rfloor$ correct guesses. Prove that no strategy can be sure to improve on this!

3. If all three hats are puce, then no student will raise a hand and each lass knows immediately that her hat is puce.

If there is just one rouge hat among the group, then two women raise their hands and each player knows immediately the color of her own hat.

If there are two rouge hats among the group, then all three players raise their hands. The students with the rouge hats know immediately the color of their own hats ("If my hat were puce, why would my rouge-topped colleague be raising her hand?"). The student with the puce hat is unsure of her color.

A potential difficulty occurs if all three hats are rouge. No player can *immediately* deduce the color of her own hat and the group will be silent. But then each woman will reason: "If my hat were puce, then someone would have spoken. My hat must therefore be rouge." All three will speak simultaneously.

Comment: It is interesting to see how, in practice, a group reacts to the situation of three rouge hats. (If players play with cards, eventually three randomly chosen cards will be red.) How long is the group willing to wait to be confident of a three-rouge situation?

Taking it further: Play and analyze the game with four players (again with rouge and puce hats). Five players?

4. If a pixie sees k rouge hats, she should announce puce at the $(100 - k)$-th minute.

Taking it further: What if the pixies weren't told that there is at least one puce hat among the group? What is the best they could hope for with regard to their survival?

5. A gnome should say "My hat is puce" if he sees no puce hats in front of him. The first gnome to say this will be correct.

Taking it further: Can the gnomes guarantee their survival if they were not sure of the presence of a puce hat?

6. The gnome at the back of the line, who sees nine hats, says "puce" if the count of puce hats he sees is odd, "rouge" otherwise. Each of the remaining nine gnomes, upon hearing the announcements made behind him and seeing the hats before him, can now correctly deduce the color of his own hat. The survival of nine out of ten gnomes is assured.

7. Number the colors 0 through 99. Working mod 100, the gnome at the back of the line announces the sum of the colors he sees. Each of the remaining nine gnomes, upon hearing the announcements made behind him and seeing the hats before him, can now correctly surmise the color of his hat. As before, the survival of nine out of ten gnomes is assured.

8. The first gnome, the one who sees 9 hats, should say rouge if he sees 9 puce hats, and pass otherwise. Each gnome thereafter should say "puce" if some gnome before him made a guess, rouge if all the gnomes who declared before him passed and he sees nothing but puce hats in front of him, and pass otherwise. The gnomes are assured survival as a group in every possible arrangement of colored hats except one: when all hats are puce. The gnomes thus have a 1023/1024 chance of survival.

Taking it further: What changes if the person placing hats on the gnomes' heads is malicious and is able to assign the colors?

9. Have each gnome make a guess only if he sees among his colleagues two hats the same color. His guess should be the color opposite of what he sees. Of the eight equally likely possibilities for hat colors among the three gnomes, the group shall survive in six of those scenarios.

10. For $N = 2^n - 1$ players, the maximal chance of survival is $N/(N + 1) = (2^n - 1)/2^n$. One attains this using the following strategy, which is closely tied to the mathematics of error-correcting codes.

Give every gnome a number from 1 to $N = 2^n - 1$ written as an n-bit string in binary. (Player 0, for instance, is assigned the string 000…0, player 1 the string 000…01, and so forth.) We'll add such strings bit-wise mod 2, that is, without carries, and denote this addition by $\oplus$. Thus, for this addition,

$$01011 \oplus 11001 = 10010.$$

(This type of addition is known as "string-wise exclusive-or" or "nim-sum".) From now on the word *sum* refers to a sum of strings added bitwise (mod 2), with each string being a vector representation of a position number. Also, note that, mod 2, we have $-1 = 1$, and so for a string k we have $-k = k$.

Have each gnome compute her *Visible Rouge Sum* (VRS), that is, the sum of those numbers whose owners she sees wearing a rouge hat. A gnome guesses rouge if her VRS is 0 (the all-zero bit string) and puce if her VRS equals her own number. Otherwise, the gnome passes. This strategy does the trick. Here's why:

First, suppose the sum of *all* numbers of rouge-hatted gnomes is 0. A puce-hatted gnome will have VRS = 0, since her own number does not contribute to the total. By the strategy, she guesses rouge—incorrectly. A rouge-hatted gnome with number k will have his VRS $= -k = k$, since his number does contribute to the total. By the strategy, he guesses puce—also incorrectly. Thus all gnomes guess and do so incorrectly. This is a losing configuration.

On the other hand, suppose the sum of *all* numbers of rouge-hatted gnomes is $j \neq 0$. If gnome j is wearing a puce hat, she sees j and guesses puce; if she is wearing a rouge hat, she sees 0 and guesses rouge. In both cases, gnome j guesses correctly. For $k \neq j$, however, player k sees $j \oplus k$ which is neither 0 nor k; and by the strategy, player k passes. Only one gnome makes a guess and does so correctly, so this is a winning configuration.

Finally, there are 2^n possible sums of all rouge-hatted gnomes. The $N = 2^n - 1$ nonzero sums lead to a win, and the one 0 sum leads to a loss. Hence, the probability of a win is exactly $N/(N + 1)$.

To see why no given strategy can do better than this, consider making a table of all 2^n possible distributions of hats and the guesses, or passes, each gnome would make in each scenario. As each gnome's response is governed only by the hats she sees, each correct guess made by a gnome in one scenario is matched by an incorrect guess in another scenario, the one in which this gnome's hat is of opposite color. Thus if there are c correct guesses distributed throughout the entire table of scenarios, there are also c incorrect guesses.

A strategy that leads to optimal chances of survival for the group "places" all the incorrect guesses among as few scenarios as possible and "spreads" out all the correct guesses through as many as possible. In fact, ideally, there would be c scenarios in which just one gnome makes a correct guess and the rest pass, no scenarios in which each gnome passes, and as few scenarios as possible with gnomes making incorrect guesses. (In fact, the least number of such scenarios we can hope for is c/N.) This gives an upper bound on the probability of the group surviving, namely:

$$\frac{c}{c + 0 + \frac{c}{N}} = \frac{N}{N+1}$$

As we have seen, for $N = 2^n - 1$, this upper bound is attainable. And it is not too difficult to show that it is attainable only for these numbers.

With c scenarios that lead to a win, we have the probability of surviving is also given by $c/2^N$. This leads to the equation $(N+1)c = N \cdot 2^N$, stating that $N \cdot 2^N$ is a multiple of $N + 1$. But since N and $N + 1$ are coprime, we must have that $N + 1$ is a factor of 2^N and so is a power of two: $N + 1 = 2^n$, for some n. That is, N must be of the form $2^n - 1$.

Taking it further: What if N is *not* one less than a power of two? (This remains an area of active research.)

Taking it even further: What if the players are allowed two, or even three, rounds of guessing?

11. The gnomes can pretend they are standing in a line and follow the strategy of question 8! If there are N gnomes, they have a $(2^N - 1)/2^N$ chance of survival (and if $N = 2^n - 1$, this equals $\left(2^{2^n-1} - 1\right)/2^{\left(2^n-1\right)}$, much closer to 1 than before!).

Taking it further: Is this the optimal chance of survival?

12. The pixies should arrange themselves in pairs and each pair should follow the strategy of question 1. This guarantees the survival of 50 pixies.

As the colors of the hats are chosen by the flip of a coin (and no information of any kind can be deduced by any individual player during play of this game) each player has precisely a 50% chance of survival. Thus the expected number of players to survive among the group is 50. One cannot guarantee the survival of more. ■

Further Reading

M. Bernstein, "The Hat Problem and Hamming Codes," *FOCUS*, 21 No. 8 (2001), 4-6. (For questions 9 and 10)

S. Butler; M.Hajiaghayi; R. Kleinberg; T. Leighton, "Hat Guessing Games," *SIAM J. of Discrete Math*, 22 No. 2 (2008) 592-605. (For questions 9, 10 and 11)

M. Gardner, *Penrose Tiles to Trapdoor Ciphers*, W.H. Freeman, New York, 1989. (For questions 3 and 4)

Akihiro Nozaki, *Anno's Hat Tricks* (illustrated by Mitsumasa Anno), Philomel Books, New York, 1985.

P. Winkler, *Mathematical Puzzles: A Connoisseur's Collection*, A.K. Peters, Natick, MA, 2004. (For questions 6, 7, and 12)