Propp's base 3/2 question

Don Reble

2016 September 27

1 Introduction

Jim Propp, posting to the `math-fun` maillist, asks

> Is it possible to write $\frac{1}{3}$ as an infinite sum of the form $a_1(\frac{2}{3})^1 + a_2(\frac{2}{3})^2 + \cdots$ where the sequence of nonnegative integer coefficients $a_1, a_2, \cdots$ is eventually periodic?

Later he adds that although the digits are nonnegative, they may be arbitrarily large.
(BTW, $1.(-1) = \frac{1}{3}$, if negative digits are allowed.)

Eventually periodic means that the digit sequence has the form $h_1 h_2 h_3 \cdots h_m \ (p_1 p_2 \cdots p_n)^*$: the subsequence $p_{1..n}$ repeats forever. The first part $h_{1..m}$ is the head H; $p_{1..n}$ is the period P.

It is traditional to put infinite sequences of digits to the *right* of the radix point. In effect, Jim asks about base $\frac{3}{2}$ representations.

2 Digit restrictions

If a number has an EPR (eventually-periodic representation) with period n, then it has an EPR, also with period n, using only digits 0,1,2.

In base $\frac{3}{2}$, $3 = 20$. If the period of an EPR has a digit greater than 2, one can subtract 3 from each corresponding digit in every period, and add 2 to the digits on their left. That does not alter the value of the EPR, but it does decrement the digit-sum of the period. (This works even if the digit-on-the-left is in a different period or in the head.) Repeat until all period digits are less than 2. The process ends because the initial digit-sum was finite. Then do it to the finite head of the EPR.

(The new EPR has period length n, but that needn't be the fundamental period length. $050505\cdots \to 222222\cdots$)

3 Digit sequence value

The point-and-digit sequence $.D = .d_1 d_2 \cdots d_z$ has value $V(.D) = \sum_{j=1}^{z} d_i (\frac{2}{3})^j$. The value of the concatenation $.HP^*$ is

$$V(.HP^*) = V(.H) + \left(\frac{2}{3}\right)^m \sum_{j=0}^{\infty} V(.P) \left(\frac{2}{3}\right)^{jn}$$

$$= V(.H) + V(.P) \left(\frac{2}{3}\right)^m \frac{3^n}{3^n - 2^n}$$

$$= V(.H) + \left(\frac{2}{3}\right)^m \frac{3^n V(.P)}{3^n - 2^n}$$

(As before, H has length m, P has length n.)
Note that $3^n V(.P)$ is an even integer.

Which values can that have? Can it be $\frac{1}{3}$?

4 Making $\frac{1}{3}$

We need consider only small-digit (0,1,2) EPRs: SDEPRs.

Suppose that $\frac{1}{3}$ has an SDEPR. There are no nonzero digits to the left of the radix point, since $\frac{1}{3} < 1$.

$$\frac{1}{3} = V(.H) + \left(\frac{2}{3}\right)^m \frac{3^n V(.P)}{3^n - 2^n}$$

Let $Q = 3^n - 2^n$, $W = 3^n V(.P)/2$ (an integer).

$$3^{m+1}\frac{1}{3} = 3^{m+1} V(.H) + 2^m \cdot 3 \cdot \frac{2W}{Q}$$

The left side and first addend are integers; so also is the second addend. Since 2 and 3 do not divide Q, Q divides W.

$0 \le W \le 2Q$. Those extrema come from the all-0 and all-2 Ps: there are no other representations of 0 nor $2Q$.

$W = Q$ comes from the all-1 representation. If there is another representation of that value, it is obtained by incrementing some digits and decrementing others. Let d_a be the leftmost altered digit. That alteration adds or subtracts (from W) $3^n(\frac{2}{3})^a/2 = 3^{n-a}2^{a-1}$, an odd multiple of 2^{a-1}. Each other alteration adds or subtracts a multiple of $3^{n-b}2^{b-1}$ (for some $b > a$), an even multiple of 2^{a-1}. The sum of alterations cannot be zero: $P = 111\cdots1$ is the only representation of Q.

It follows that since Q divides W, P has fundamental period length 1.

> The reader may observe that this applies to any rational value $\frac{x}{3}$. For examples, $\frac{2}{3} = .1(0)^*$; $\frac{3}{3} = 1.(0)^*$. It remains to show that $\frac{1}{3}$ has no period-1 SDEPR.

$V(.111111\cdots) = 2$. If $\frac{1}{3}$ has a period-1 SDEPR, then the tail value is 0, 2, or 4 (an integer) times $(\frac{2}{3})^m$. In any case, one can add the tail value into the head, and remove the tail. And so $\frac{1}{3}$ has a finite representation, which as before can be converted to a small-digit representation $R = .d_1d_2\cdots d_z$; d_z is the last non-zero digit.

For any such representation R, $3^{z-1}R$ is not an integer, because of that last digit. But $3^{z-1}\frac{1}{3}$ is an integer, unless $z = 1$. So if $\frac{1}{3}$ has a representation, it is a multiple of .1; that is, a multiple of $\frac{2}{3}$. No!