24 Muffins, 25 Students
By William Gasarch, Answering a Question Posed by James Propp

1 A Procedure with Smallest Piece $\frac{8}{25}$

1. Divide 6 muffins $\{\frac{12}{25}, \frac{13}{25}\}$.
2. Divide 6 muffins $\{\frac{11}{25}, \frac{14}{25}\}$.
3. Divide 6 muffins $\{\frac{10}{25}, \frac{15}{25}\}$.
4. Divide 6 muffins $\{\frac{8}{25}, \frac{8}{25}, \frac{9}{25}\}$.
5. Give 3 students $\{\frac{12}{25}, \frac{12}{25}\}$.
6. Give 6 students $\{\frac{11}{25}, \frac{13}{25}\}$.
7. Give 6 students $\{\frac{10}{25}, \frac{14}{25}\}$.
8. Give 6 students $\{\frac{9}{25}, \frac{15}{25}\}$.
9. Give 4 students $\{\frac{8}{25}, \frac{8}{25}, \frac{8}{25}\}$.

2 There is no Procedure with Smallest Piece $> \frac{8}{25}$

Assume there is a procedure for 24 muffins, 25 people where everyone gets $\frac{24}{25}$.
Case 1: Some muffin is cut into ≥ 4 pieces. Then some piece is $\leq \frac{1}{4} < \frac{8}{25}$.
Case 2: Some muffin is uncut. Whoever gets that muffin has $1 > \frac{24}{25}$ which is impossible.
Case 3: Some student gets ≥ 3 pieces. One of the pieces must be $\leq \frac{24}{25} \times \frac{1}{3} = \frac{8}{25}$.
Case 4: Some student gets 1 piece. That piece must be of size $\frac{24}{25}$. Look at the muffin it came from. The other pieces add up to $1 - \frac{24}{25} = \frac{1}{25} < \frac{8}{25}$. Hence some piece is $< \frac{8}{25}$.
Case 5: All muffins are cut into either 2 or 3 pieces and all students get 2 pieces. We will call muffins cut into 2 pieces *2-muffins* and muffins cut into 3-pieces *3-muffins*. We will call the pieces from a 2-muffin *2-pieces* and the pieces from a 3-muffin *3-pieces*.
Case 5a: Some 3-piece is $> \frac{9}{25}$. Look at the muffin this piece came from. The other 2 pieces of it add up to $< 1 - \frac{9}{25} = \frac{16}{25}$. Hence some piece is $< \frac{9}{25}$.
Case 5b: Some 2-piece is $< \frac{9}{25}$. Look at the muffin this piece came from. The other piece of it is $> 1 - \frac{9}{25} = \frac{16}{25}$. Look at who gets that piece. The rest of what he gets is of size $< \frac{24}{25} - \frac{16}{25} = \frac{8}{25}$.
Case 5c: Some 2-piece is $> \frac{16}{25}$. Look at the person who has that piece. The other piece they have is of size $< \frac{24}{25} - \frac{16}{25} = \frac{8}{25}$.

OKAY, we now got rid of all the easy cases. Now what?

The following diagram illustrates what is going on:

$$\underset{\frac{8}{25}}{(} \quad \text{3-pieces} \quad \underset{\frac{9}{25}}{)(} \quad \text{2-pieces} \quad \underset{\frac{16}{25}}{)}$$

Let m_2 (m_3) be the number of 2-muffins (3-muffins). Since (1) every student gets 2 pieces there are 50 pieces, and (2) there are 24 muffins:

$$
\begin{aligned}
2m_2 + 3m_3 &= 50 \\
m_2 + m_3 &= 24
\end{aligned}
$$

Hence $m_2 = 22$ and $m_3 = 2$, so there are 44 2-pieces and 6 3-pieces

$$
\underset{\frac{8}{25}}{(} \quad \text{6 3-pieces} \quad \underset{\frac{9}{25}}{)(} \quad \text{44 2-pieces} \quad \underset{\frac{16}{25}}{)}
$$

$$\left(\frac{8}{25}, \frac{9}{25}\right) \text{ has 6 pieces.}$$

There are 6 muffins in $(\frac{8}{25}, \frac{9}{25})$. We have a bijection from these 6 pieces to the pieces in $(\frac{15}{25}, \frac{16}{25})$ as follows: if $x \in (\frac{8}{25}, \frac{9}{25})$ then let Alice have x. Alice has only two pieces. Map x to y, the other piece Alice has. Since $x + y = \frac{24}{25}$ and $x \in (\frac{8}{25}, \frac{9}{25})$, $y \in (\frac{15}{25}, \frac{16}{25})$. The map is invertible. Hence

$$\left(\frac{15}{25}, \frac{16}{25}\right) \text{ has 6 pieces.}$$

We have a bijection from the pieces in $(\frac{15}{25}, \frac{16}{25})$ to the pieces in $(\frac{9}{25}, \frac{10}{25})$. Let x be a piece in $(\frac{15}{25}, \frac{16}{25})$. Look at the muffin that x came from. Since x is a 2-piece there is only one other piece from that muffin. Call its size y. Then $x + y = 1$. Since $x \in (\frac{15}{25}, \frac{16}{25})$ and $x + y = 1$, $y \in \frac{9}{25}, \frac{10}{25})$. The map is invertible. Hence

$$\left(\frac{9}{25}, \frac{10}{25}\right) \text{ has 6 pieces.}$$

By similar methods to the two above we can show that the following open intervals have 6 pieces:

$$\left(\frac{14}{25}, \frac{15}{25}\right), \left(\frac{10}{25}, \frac{11}{25}\right) \left(\frac{13}{25}, \frac{14}{25}\right)$$

$$\left(\frac{11}{25}, \frac{12}{25}\right), \left(\frac{12}{25}, \frac{13}{25}\right)$$

There are no pieces of size the endpoint of any of these intervals. Since these 7 intervals are all distinct and have 6 pieces in them, there are 42 pieces. This contradicts that we have 50 pieces. Hence this case cannot occur.