Negative Numbers in Combinatorics: Geometrical and Algebraic Perspectives

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Slides for this talk are on-line at

http://jamespropp.org/msri-up12.pdf

I. Equal combinatorial rights for negative numbers?

Counting

If a set S has n elements, the number of subsets of S of size k equals

$$n(n-1)(n-2)\cdots(n-k+1)/k!$$

Let's take this formula to be our definition of $\binom{n}{k}$.

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Examples:

$n=4$: $\binom{4}{3} = 4 \cdot 3 \cdot 2/6 = 4$

$n=3$: $\binom{3}{3} = 3 \cdot 2 \cdot 1/6 = 1$

$n=2$: $\binom{2}{3} = 2 \cdot 1 \cdot 0/6 = 0$

$n=1$: $\binom{1}{3} = 1 \cdot 0 \cdot (-1)/6 = 0$

$n=0$: $\binom{0}{3} = 0 \cdot (-1) \cdot (-2)/6 = 0$

Extrapolating

If there were such a thing as a set with -1 elements, how many subsets of size 3 would it have?

One commonsense answer is "Zero, because a set of size < 3 can't have any subsets of size 3!"
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$n = -1$: $\binom{-1}{3} = (-1) \cdot (-2) \cdot (-3)/6 = -1$

Likewise, if there were such a thing as a set with -2 elements, how many subsets of size 3 would it have, according to the formula?

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What might this mean?

II. Hybrid sets

Loeb, following Stanley (see the reading list at the end of the slides), defines a "set with a negative number of elements" as a set whose elements can be selected more than once, and indeed as many times as one likes (multisets with unbounded multiplicity).

Example: Let $S = \{\{x, y\}\}$ where x and y are "negative elements". The subsets of S of size 2 are $\{\{x, x\}\}$, $\{\{x, y\}\}$, and $\{\{y, y\}\}$.

Check: $\binom{-2}{2} = (-2)(-3)/2! = 3$.

Trouble (and trouble averted)

The subsets of S of size 3 are $\{\{x, x, x\}\}$, $\{\{x, x, y\}\}$, $\{\{x, y, y\}\}$, and $\{\{y, y, y\}\}$. There are 4 such subsets.

Compare: $\binom{-2}{3} = (-2)(-3)(-4)/6 = -4$.

Why the minus sign?

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Decree that a multiset with k negative elements (some of which may be equal to each other) has **weight** $(-1)^k$.

Theorem: If S has m negative elements, $\binom{-m}{k}$ is the sum of the weights of all the k-element subsets of S, where elements can be repeated.

Proof

All the k-element subsets have weight $(-1)^k$, so it's enough to show that $\binom{-m}{k}$ equals $(-1)^k$ times the number of such subsets.

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Evaluate $\binom{-m}{k}$:

$$\begin{aligned}\binom{-m}{k} &= (-m)(-m-1)\cdots(-m-k+1)/k! \\ &= (-1)^k(m)(m+1)\cdots(m+k-1)/k! \\ &= (-1)^k\binom{m+k-1}{k}.\end{aligned}$$

Proof (concluded)

Count the subsets using “stars and bars”:

$$\begin{array}{ccc} x & x & x \\ x & x & y \\ x & y & y \\ y & y & y \end{array} \qquad\qquad \begin{array}{cccc} * & * & * & | \\ * & * & | & * \\ * & | & * & * \\ | & * & * & * \end{array}$$

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There are k stars and $m-1$ bars, so there are $\binom{k+m-1}{k}$ permutations.

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Since $\binom{m+k-1}{k} = \binom{k+m-1}{k}$, we are done.

Reciprocity law

The formula

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}$$

is valid for both positive and negative values of m.

We call it a (combinatorial) **reciprocity law**, because it relates the values of $\binom{n}{k}$ for the two different values of n whose sum is some specified number (in this case the number $k-1$).

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Negative elements contribute -1 (multiplicatively) to the weight of a subset they belong to, and they can be chosen repeatedly.

An example

Let S be the hybrid set with 1 ordinary element, a, and 2 negative elements, x and y, so that the “number of elements of S according to sign” is $1 - 2 = -1$.

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Subsets of weight $+1$:
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Total weight of all subsets: $(4)(-1) + (3)(+1) = -1$.

Compare: $\binom{-1}{3} = (-1)(-2)(-3)/6 = -1$.

“Homework”

Prove that if a hybrid set S has A ordinary elements and B negative elements, then the sum of the weights of the k-element subsets of S is $\binom{A-B}{k}$.

The bigger picture

This last result quells some of our anxieties about negative elements but it leaves us wondering what all this is good for, and how it relates to the rest of mathematics.

And we're also left wondering how one could have come up with the idea of negative elements or hybrid sets, and how one can come up with analogous ideas for other attempts to extend combinatorics to the negative integers.

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Example: the two-sided Fibonacci sequence

$$\ldots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, \ldots$$

Ordinary Fibonacci numbers have lots of combinatorial interpretations. But what do the terms to the left of the 0 "count"?

Things to come

The rest of the talk:

- Use Euler characteristic to give a geometric interpretation of $\binom{-2}{3}$.
- Use recurrences to derive directly an implicit combinatorial meaning for $\binom{-2}{3}$.
- Apply the ideas to the two-sided sequence of Fibonacci numbers.
- Indicate prospects for future work.

III. Euler characteristic

One dimension

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$\chi([0,1]) = \chi(\{0\} \cup (0,\frac{1}{2}) \cup \{\frac{1}{2}\} \cup (\frac{1}{2},1) \cup \{1\}) = 1-1+1-1+1 = 1$

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$\chi(\cdot)$ is called **Euler measure** or **(combinatorial) Euler characteristic**.

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Write S as a disjoint union of V vertices, E edges, F faces, ...

Then

$$\chi(S) := V - E + F - \dots$$

is independent of the decomposition of S into vertices, edges, faces, etc.

Examples of Euler characteristic

Let $S = \{(x, y) \in \mathbf{R}^2 : \; 0 < x < y < 1\}$.

S is an open triangle containing 0 vertices, 0 edges, and 1 face, so $\chi(S) = 0 - 0 + 1 = 1$.

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Similarly, let $S = \{(x, y, z) \in \mathbf{R}^3 : \; 0 < x < y < z < 1\}$.

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More generally, a bounded open (resp. bounded closed) polyhedral subset of $\mathbf{R}^d$ has Euler measure $(-1)^d = \pm 1$ (resp. $(+1)^d = 1$).

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χ is **homeomorphism-invariant**:
If two polyhedral subsets are homeomorphic, they have the same Euler measure.
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χ is **multiplicative**:

$$\chi(S_1 \times S_2) = \chi(S_1)\chi(S_2).$$

(Ordinary Euler characteristic does not have any of these properties.)

Combinatorial Euler characteristic and binomial coefficients

Theorem (McMullen? Morelli?): If $S \subseteq \mathbf{R}$ with

$$\chi(S) = m \in \mathbf{Z},$$

and we define

$$\binom{S}{k} := \{(x_1, x_2, \ldots, x_k) \in S^k \subseteq \mathbf{R}^k : \; x_1 < x_2 < \cdots < x_k\},$$

then

$$\chi(\binom{S}{k}) = \binom{m}{k}.$$

The geometrical meaning of $\binom{-2}{3}$

Example: $S = (0,1) \cup (2,3)$. $\binom{S}{3}$ is the disjoint union of four open tetrahedra: $\{(x,y,z) : 0 < x < y < z < 1\}$,
$\{(x,y,z) : 0 < x < y < 1,\ 2 < z < 3\}$,
$\{(x,y,z) : 0 < x < 1,\ 2 < y < z < 3\}$,
$\{(x,y,z) : 2 < x < y < z < 3\}$,
each of which has Euler measure $(-1)^3$, so

$$\begin{aligned} \chi(\binom{S}{3}) &= (-1) + (-1) + (-1) + (-1) \\ &= -4 = \binom{-2}{3} = \binom{\chi(S)}{3}. \end{aligned}$$

Hybrid sets, in a geometrical setting

Likewise, you should check that with $S = (0,1) \cup (2,3) \cup \{4\}$,

$$\begin{aligned}
\chi(\binom{S}{3}) &= (-1) + (-1) + (-1) + (-1) + (1) + (1) + (1) \\
&= -1 = \binom{-1}{3} = \binom{1-2}{3} = \binom{\chi(S)}{3}.
\end{aligned}$$

IV. Encoding combinatorics algebraically

Symmetric polynomials

The algebraic analogue of $\binom{4}{3}$ is

$$e_3^{(4)} := x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.$$

The algebraic analogue of $\binom{3}{3}$ is

$$e_3^{(3)} := x_1x_2x_3.$$

These are **symmetric polynomials**: they are unaffected by swapping any two variables.

More generally, $e_n^{(m)}$ (the "elementary symmetric function of degree n in m variables") is the sum of all products of the variables $x_1, \ldots, x_m$, taken n at a time.

Recurrences

Symmetric functions satisfy linear recurrences:

$$\begin{aligned} e_3^{(4)} &= (x_1x_2x_3) + (x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\ &= (x_1x_2x_3) + (x_1x_2 + x_1x_3 + x_2x_3)x_4 \\ &= e_3^{(3)} + x_4\, e_2^{(3)} \end{aligned}$$

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More generally,

$$e_3^{(n)} = e_3^{(n-1)} + x_n\, e_2^{(n-1)}$$

Recurrences in reverse

To discover the right definition of $e_3^{(n)}$ with $n < 0$, run the recurrence relation in reverse.

$$e_3^{(n)} = e_3^{(n-1)} + x_n e_2^{(n-1)}$$

becomes

$$e_3^{(n-1)} = e_3^{(n)} - x_n e_2^{(n-1)}$$

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To apply this recurrence to compute $e_3^{(n-1)}$ for all $n \in \mathbf{Z}$, we need to know $e_2^{(n-1)}$ for all $n \in \mathbf{Z}$.

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To apply this recurrence to compute $e_3^{(n-1)}$ for all $n \in \mathbf{Z}$, we need to know $e_2^{(n-1)}$ for all $n \in \mathbf{Z}$.

But we can use the same trick with e_2, reducing it to a problem of determining e_1.

Try it!

e_1:

$x_1 + x_2 + x_3, x_1 + x_2, x_1,$

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$x_1 + x_2 + x_3, x_1 + x_2, x_1, 0,$

Try it!

e_1:

$x_1 + x_2 + x_3, x_1 + x_2, x_1, 0, -x_0,$

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$x_1 + x_2 + x_3, x_1 + x_2, x_1, 0, -x_0, -x_0 - x_{-1}, -x_0 - x_{-1} - x_{-2}, \ldots$

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e_1:

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e_2:

$x_1x_2 + x_1x_3 + x_2x_3, x_1x_2,$

Try it!

e_1:
$x_1 + x_2 + x_3, x_1 + x_2, x_1, 0, -x_0, -x_0 - x_{-1}, -x_0 - x_{-1} - x_{-2}, \ldots$

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e_1:
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Try it!

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e_2:
$x_1 x_2 + x_1 x_3 + x_2 x_3, x_1 x_2, 0, 0, x_0^2, x_0^2 + x_0 x_{-1} + x_{-1}^2, \ldots$

e_3:
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e_2:
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Try it!

e_1:
$x_1+x_2+x_3, x_1+x_2, x_1, 0, -x_0, -x_0-x_{-1}, -x_0-x_{-1}-x_{-2}, \ldots$

e_2:
$x_1x_2+x_1x_3+x_2x_3, x_1x_2, 0, 0, x_0^2, x_0^2+x_0x_{-1}+x_{-1}^2, \ldots$

e_3:
$x_1x_2x_3, 0, 0, 0,$

Try it!

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$x_1 + x_2 + x_3, x_1 + x_2, x_1, 0, -x_0, -x_0 - x_{-1}, -x_0 - x_{-1} - x_{-2}, \ldots$

e_2:
$x_1x_2 + x_1x_3 + x_2x_3, x_1x_2, 0, 0, x_0^2, x_0^2 + x_0x_{-1} + x_{-1}^2, \ldots$

e_3:
$x_1x_2x_3, 0, 0, 0, -x_0^3,$

Try it!

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$x_1x_2 + x_1x_3 + x_2x_3, x_1x_2, 0, 0, x_0^2, x_0^2 + x_0x_{-1} + x_{-1}^2, \ldots$

e_3:
$x_1x_2x_3, 0, 0, 0, -x_0^3, -x_0^3 - x_0^2x_{-1} - x_0x_{-1}^2 - x_{-1}^3, \ldots$

Try it!

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$x_1x_2x_3, 0, 0, 0, -x_0^3, -x_0^3 - x_0^2x_{-1} - x_0x_{-1}^2 - x_{-1}^3, \dots$

Up to sign, we are getting a reciprocity between the elementary symmetric polynomials and the **complete homogeneous symmetric polynomials** $h_n^{(m)}$.

Unification

You should take the time to see how the formal expressions that the backward recurrence gives us (complete homogeneous symmetric polynomials) correspond to multisets, and that the behavior of the signs is compatible with the combinatorics of Loeb's negative sets.

You should also take the time to see how, for any 1-dimensional polyhedral set S in $\mathbf{R}$, the polyhedral set $\binom{S}{k}$ splits into cells, and how the monomial terms in a complete homogeneous symmetric polynomial correspond to the cells.

V. Fibonacci numbers

A combinatorial problem

In how many ways can we write n as an ordered sum of 1's and 2's?
$n = 1$: 1 way (1)
$n = 2$: 2 ways $(2,\ 1+1)$
$n = 3$: 3 ways $(1+2,\ 2+1,\ 1+1+1)$
$n = 4$: 5 ways $(2+2,\ 1+1+2,\ 1+2+1,\ 2+1+1,\ 1+1+1+1)$

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$n = 4$: 5 ways ($2 + 2$, $1 + 1 + 2$, $1 + 2 + 1$, $2 + 1 + 1$, $1 + 1 + 1 + 1$)

View each sum as a way of tiling a segment of length n by segments of length 1 and length 2, and represent each tiling by the monomial $\prod x_i$ where i varies over all locations at which there is a segment of length 1.

E.g., the 5 ordered sums with $n = 4$ correspond to the respective monomials 1, x_1x_2, x_1x_4, x_3x_4, $x_1x_2x_3x_4$.

From Fibonacci numbers to Fibonacci polynomials

Let $P_n(x_1, x_2, \dots)$ be the sum of all monomials in $x_1, \dots, x_n$ associated with tilings.

$P_1 = x_1$

$P_2 = 1 + x_1 x_2$

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From Fibonacci numbers to Fibonacci polynomials

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In general,

$$P_{n+1} = P_{n-1} + x_{n+1}P_n.$$

(From this it is easy to check that the number of terms in the polynomials P_i are the successive Fibonacci numbers: set $x_1 = x_2 = \cdots = 1$.)

Running backwards

Iterating the reverse recurrence

$$P_{n-1} = P_{n+1} - x_{n+1} P_n$$

we get
$P_0 = 1$
$P_{-1} = 0$
$P_{-2} = 1$
$P_{-3} = -x_{-1}$
$P_{-4} = 1 + x_{-1}x_{-2}$
$P_{-5} = -x_{-1} - x_{-3} - x_{-1}x_{-2}x_{-3}$
$P_{-6} = 1 + x_{-1}x_{-2} + x_{-1}x_{-4} + x_{-3}x_{-4} + x_{-1}x_{-2}x_{-3}x_{-4}$

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Theorem (Propp): For $n \geq 0$, P_{-n} is $(-1)^n$ times the polynomial obtained from P_{n-2} by replacing x_k by x_{-k} for all $k \geq 1$.

Bringing in some geometry

Fix a polyhedral set $P \subseteq \mathbf{R}$.

Call a finite subset $S \subseteq P$ **fabulous** if for all $t, t' \in S \cup \{+\infty, -\infty\}$, $\chi((P \setminus S) \cap (t, t'))$ is even.

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E.g.: the fabulous subsets of $\{1, 2, 3\}$ are $\{1\}$, $\{3\}$, and $\{1, 2, 3\}$.

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E.g.: The fabulous subsets of $(0, 1) \cup (2, 3) \cup (4, 5) \cup (6, 7)$ are the empty set and all sets of the form $\{x, y\}$ with $2 < x < 3$, $4 < y < 5$.

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Theorem (Propp): If $P \subseteq \mathbf{R}$ with $\chi(P) = n$, the set of fabulous subsets of P has Euler characteristic equal to the Fibonacci number $(\phi^{n+1} - \phi^{-n-1})/\sqrt{5}$ (with $\phi = (1 + \sqrt{5})/2$).

VI.What's next?

A curriculum of combinatorics

Say a combinatorial sequence $s_1, s_2, \ldots$ is of **grade** d if $s_n \sim c^{n^d}$ in the sense that for all $c > 1$ and all $\epsilon > 0$, $s_n / c^{n^{d+\epsilon}} \to 0$ and $s_n / c^{n^{d-\epsilon}} \to \infty$.

A curriculum of combinatorics

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"Grade 0" (or "kindergarten") combinatorics: n^2, $\binom{n}{2}$, $\binom{n}{3}$, ...

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"Grade 1" combinatorics: 2^n, $(\phi^{n+1} - \phi^{-n-1})/\sqrt{5}$ (Fibonacci numbers), $\frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers), $n!$, n^n, $\ldots$

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"Grade 1" combinatorics: 2^n, $(\phi^{n+1} - \phi^{-n-1})/\sqrt{5}$ (Fibonacci numbers), $\frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers), $n!$, n^n, ...

"Grade 2" combinatorics: $2^{n(n+1)/2}$ (number of perfect matchings of the Aztec diamond graph of order n), $\prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{k=1}^{n} \frac{i+j+k-1}{i+j+k-2}$ (number of rhombus tilings of the regular hexagon of order n), $\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$ (number of alternating-sign matrices of order n), Somos sequences, ...

The rampancy of reciprocity

In very many cases, one finds that a one-sided sequence $s_1, s_2, \ldots$ with enumerative meaning admits a natural extension to a two-sided sequence $\ldots, s_{-1}, s_0, s_1, \ldots$; and in many of those cases, one finds that the resulting two-sided sequence satisfies a combinatorial reciprocity property (i.e., satisfies $s_n = \pm s_{a-n}$ for all n, for some a).

Why?

Finding a setting in which the numbers $s_{-1}, s_{-2}, \ldots$ actually mean something is one way to try to resolve the mystery of combinatorial reciprocity.

To learn more, read:

Negative sets have Euler characteristic and dimension, by Stephen Schanuel; Proceedings of Category Theory, 1990, Lecture Notes in Mathematics vol. 1488, pp. 379–385.

Sets with a negative number of elements, by Daniel Loeb; *Advances in Mathematics* **91** (1992), 64–74; http://jamespropp.org/negative.pdf

Euler Measure as Generalized Cardinality, by James Propp; http://front.math.ucdavis.edu/0203.5289

Exponentiation and Euler measure, by James Propp; *Algebra Universalis* **49**, no. 4, 459–471 (2003); http://front.math.ucdavis.edu/0204.5009

And also

A Reciprocity Theorem for Monomer-Dimer Coverings, by Nick Anzalone and John Baldwin and Ilya Bronshtein and T. Kyle Petersen; *Discrete Mathematics and Theoretical Computer Science* **AB(DMCS)**, 2003, 179–194; http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAB0115

A Reciprocity Theorem for Domino Tilings, by James Propp; *Electronic Journal of Combinatorics* **8**, no. 1, R18 (2001); http://www.combinatorics.org/ojs/index.php/eljc/article/view/v8i1r18

A Reciprocity Sequence for Perfect Matchings of Linearly Growing Graphs, by David Speyer (unpublished); http://www.math.lsa.umich.edu/~speyer/TransferMatrices.pdf

Combinatorial Reciprocity Theorems, by Richard Stanley; *Advances in Mathematics* **14**, 1974, 194–253; www-math.mit.edu/~rstan/pubs/pubfiles/22.pdf

Thanks for listening!

The slides for this talk are on-line at

http://jamespropp.org/msri-up12.pdf