A Pentagonal Number Theorem for Tribone Tilings

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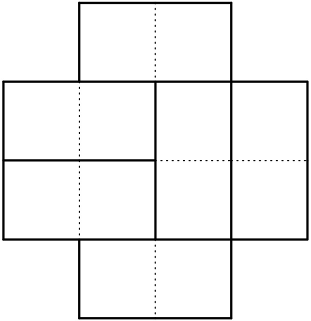
(based on joint work with Jesse Kim, UCSD)

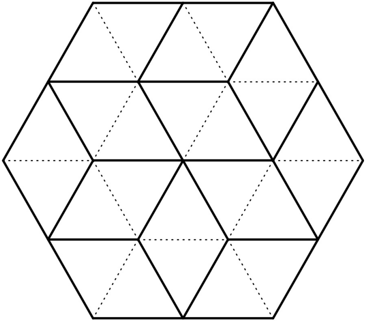
Michigan State University Combinatorics Seminar
November 2, 2022

Slides at http://jamespropp.org/msu22.pdf
Video at . . . ...

Overview

There's a well-developed theory of enumeration of domino tilings of polyominos and diamond tilings of polyiamonds.





What else with a similar flavor is out there waiting to be studied?

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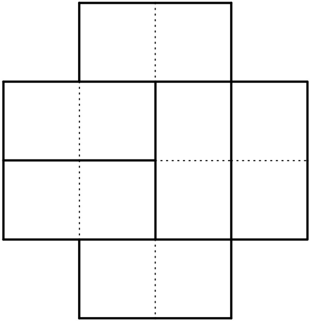
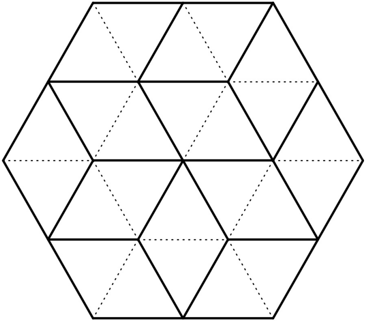


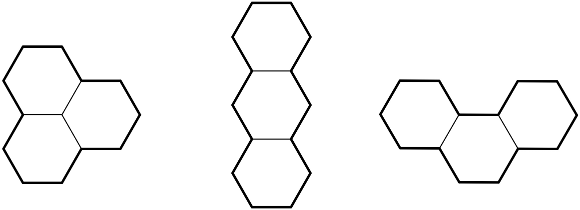


What else with a similar flavor is out there waiting to be studied?

Trihex tilings of polyhexes!

Trihexes

In a honeycomb grid there are three ways to join three hexagons to form a connected polygon. Nobody has given them coordinated names, so I've called them the stone, the bone, and the phone. ("Bone" is a shortening of "tribone", Thurston's name for this kind of tile; I chose "stone" to rhyme with "bone", and Timothy Chow suggested "phone" for the third kind of trihex.)

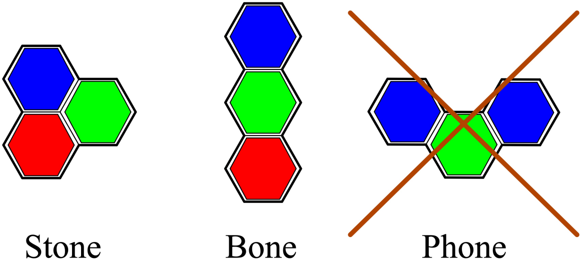


Stone Bone Phone

("Bone" is sometimes used as a synonym for "domino".)

Stones and bones

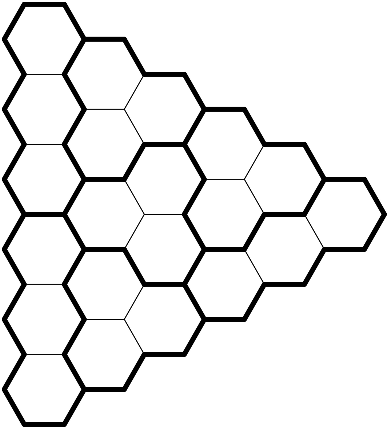
There is a natural way to 3-color the cells in the honeycomb grid (unique up to permutation of colors). Under this coloring, each stone and each bone consists of one hexagonal cell of each color, just as the square grid can be 2-colored (using the usual checkerboard coloring) so that every domino consists of one square cell of each color.



But phones don't have this property, so we'll forbid them.

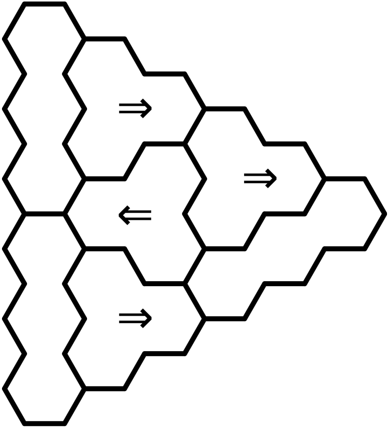
Conway and Lagarias

Tilings of regions by stones and bones were first seriously studied by Conway, Lagarias, and Thurston in the 1980s.



Conway and Lagarias

To explain their main result, we'll need to distinguish between right-pointing stones and left-pointing stones.



Conway and Lagarias

Conway and Lagarias showed that triangles like this can be tiled by stones alone provided the side-length ($n = 6$ in the example) satisfies a certain mod-12 condition, and that such triangles can never be tiled by bones alone.

A major tool in their proof was a theorem showing that if a region R can be tiled by stones and bones, the number of right-pointing stones minus the number of left-pointing stones is independent of the tiling; it depends on the region R alone.

Even more interesting than the result was the method: combinatorial group theory.

Thurston

Thurston ("Conway's Tiling Groups") rephrased the method using cell complexes and combinatorial topology.

A few years later I ("A pedestrian approach to a method of Conway, or, A tale of two cities"; password is `aztec`) presented the method in a more low-tech way, using what I called "shadow paths" (implicit but not emphasized in Thurston's paper). It's ad hoc but more accessible. Instead of doing a straight rewrite of Thurston, I applied the shadow path method to a problem about tetrominoes.

Today I'll show how shadow paths can be applied in the original setting of the result of Conway and Lagarias (before applying the method to prove something new).

Into the shadows

We'll shadow the 1-skeletons of tilings in a weird way that can turn boundaries of tiles into self-intersecting paths. By measuring the algebraic area "enclosed" by the shadows we learn things about the original tiling.

The **algebraic area** associated with a plane curve is a signed quantity: it's positive if the curve encloses a region counterclockwise, it's negative if the curve encloses a region clockwise, and it could be zero if the curve crosses itself (e.g., a symmetrical figure-eight curve encloses signed area zero).

For a curve in a grid, it's the sum, over all grid-cells C, of the winding-number of the curve around C.

Even and odd vertices

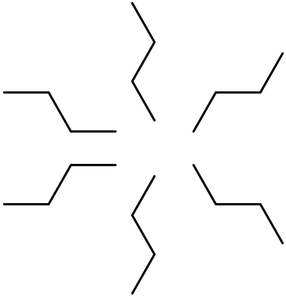
Call vertices in the hexagon grid **even** if they have edges emanating at 0, 120, and 240 degrees and **odd** if they have edges emanating at 60, 180, and 300 degrees.

Even vertex   Odd vertex

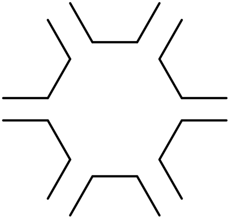
Note that a path in the honeycomb grid alternates between even vertices and odd vertices.

Weaving and winding

Consider three consecutive edges e, e', e'' in a path that travels along edges in the honeycomb grid. We say that this part of the path is **weaving** if e and e'' are parallel and **winding** otherwise.



Weaving



Winding

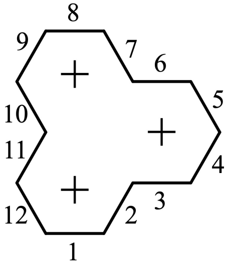
Shadowing a path

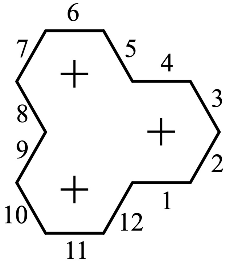
We say a path P' in the 1-skeleton of the honeycomb grid **shadows** another such path P of the same length if for all i the ith vertex of P' has the same parity (even vs. odd) as the ith vertex of P, and P' weaves where P winds and vice versa.

The shadow of a general closed path can be a non-closed path and vice versa, but if P is the (counterclockwise) boundary of a stone or bone (necessarily a closed path), and P' shadows P, then P' is closed as well.

Shadowing a right-pointing stone

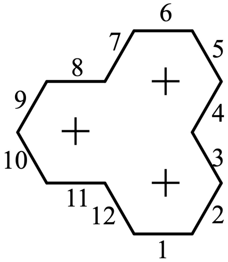
The shadow of the counterclockwise boundary of a right-pointing stone is the counterclockwise boundary of a right-pointing stone, enclosing algebraic area $+3$ (measured in hexagons).

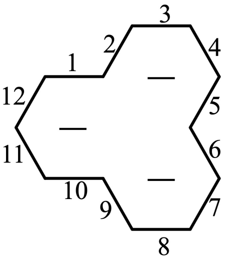




Shadowing a left-pointing stone

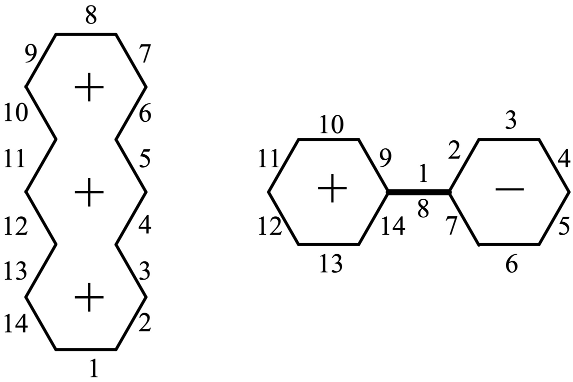
The shadow of the counterclockwise boundary of a left-pointing stone is the **clockwise** boundary of a left-pointing stone, enclosing algebraic area -3 (measured in hexagons).





Shadowing a bone

The shadow of the counterclockwise boundary of a bone is a self-intersecting "barbell" path that encircles one hexagonal cell in a counterclockwise direction and one hexagonal cell in a clockwise direction, thus enclosing algebraic area zero.



Shadowing a tiling

The shadows of the boundaries of the cells in a stones-and-bones tiling of a region R fit together to give a tangle of stones and barbells.

The sum of the signed areas of the shadows of the tiles is $3n_R - 3n_L + 0n_B$ where n_R is the number of right-pointing stones in the tiling, n_L is the number of left-pointing stones in the tiling, and n_B is the number of bones in the tiling.

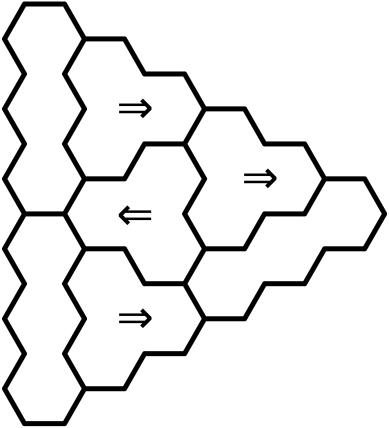
But the sum of the signed areas of the shadows of the tiles also equals the signed area enclosed by the shadow of the boundary of R, which doesn't depend on the tiling.

So $3n_R - 3n_L + 0n_B$ is independent of the tiling.

The Conway-Lagarias invariant

$n_R - n_L$ (or $3(n_R - n_L)$) is the Conway-Lagarias invariant of R.

To show that a region R cannot be tiled by bones alone, it's enough to show that its Conway invariant is nonzero. And to do that, it's enough to exhibit **one** tiling with $n_R \neq n_L$, e.g.:



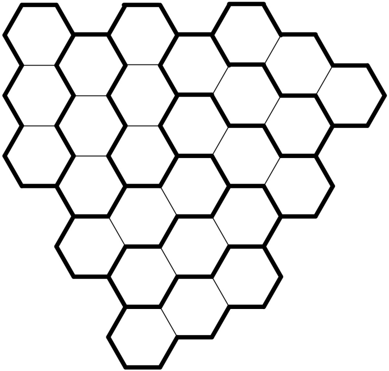
The Conway-Lagarias invariant

But this trick isn't always helpful for handling infinite families of regions, e.g., the two-parameter family of regions I call **benzels**.

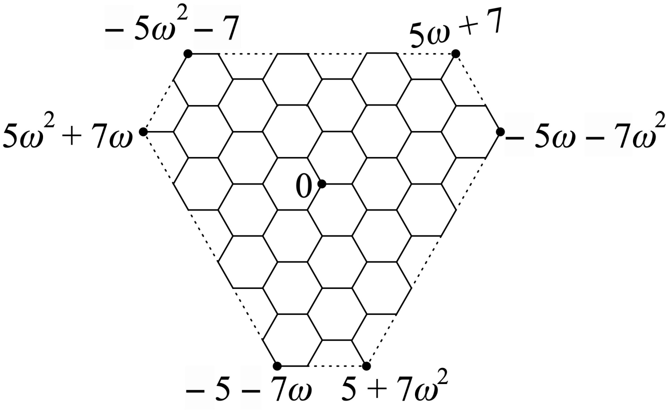
Something more algebraic is needed.

Benzels

Here's an example of a benzel, specifically the (5,7)-benzel:



Benzels


$-5\omega^2 - 7$
$5\omega + 7$
$5\omega^2 + 7\omega$
$-5\omega - 7\omega^2$
0
$-5 - 7\omega$
$5 + 7\omega^2$

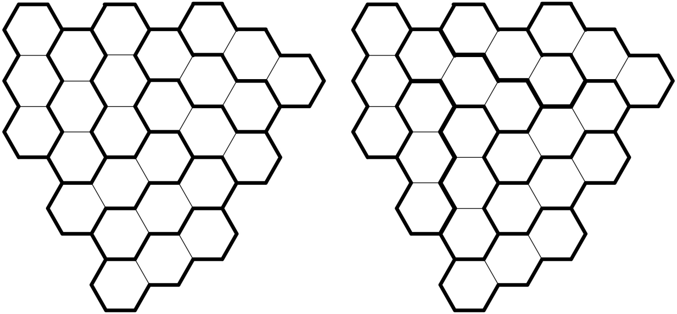
Benzels

The (a, b)-benzel is what we get when we replace 5 and 7 in the diagram by a and b respectively, for integers a, b satisfying $2 \leq a \leq 2b$ and $2 \leq b \leq 2a$, retaining only those hexagonal cells that lie fully inside the bounding hexagon.

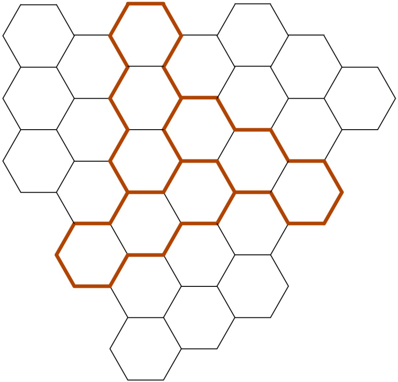
What justifies the definition are the nice (mostly still unproved) formulas that arise; see "Trimer covers in the triangular grid" and "Trimer covers in the triangular grid: twenty mostly open problems".

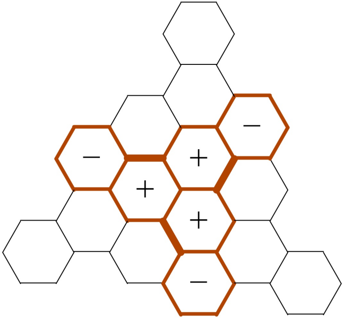
Bones tilings of the (5,7)-benzel

As we'll see most benzels can't be tiled by bones alone, but the (5,7)-benzel can, indeed in two different ways:

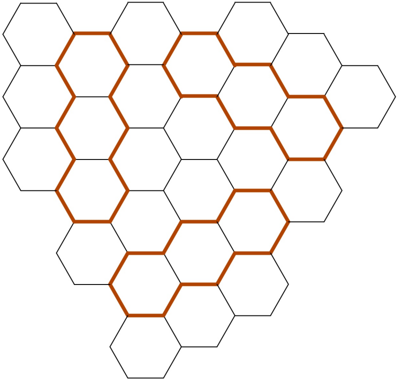


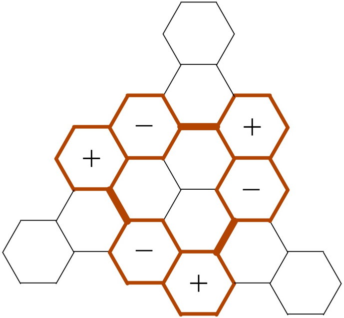
Tiling 1: Inner tiles



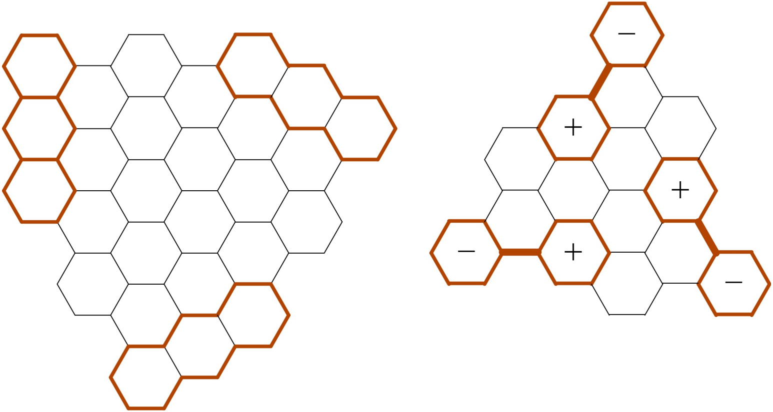


Tiling 1: Intermediate tiles

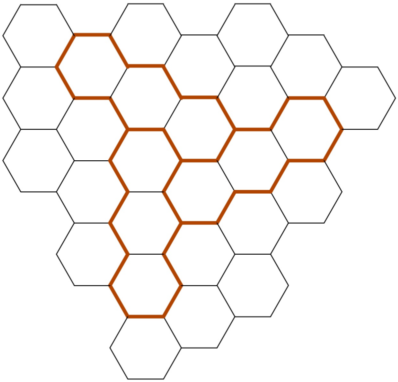


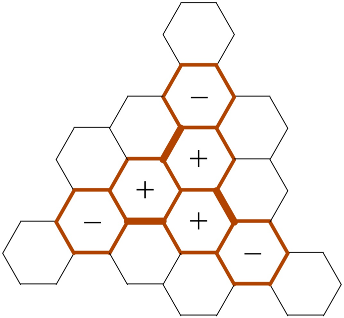


Tiling 1: Outer tiles

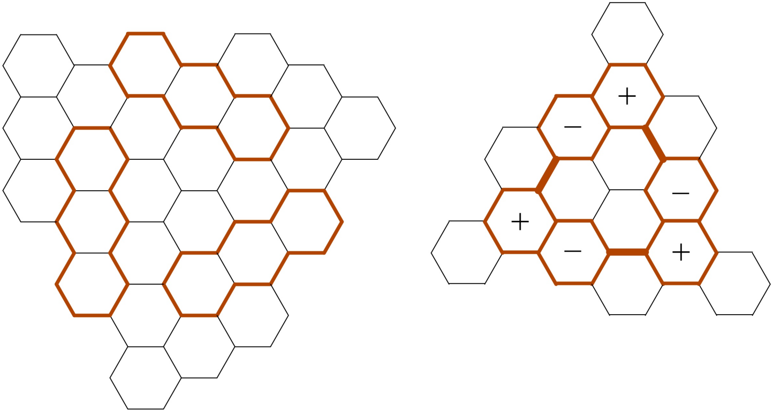


Tiling 2: Inner tiles





Tiling 2: Intermediate tiles



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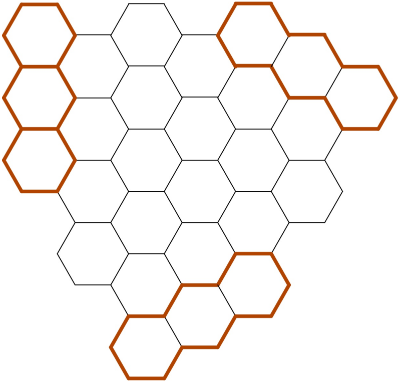
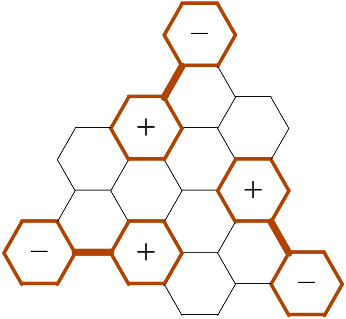
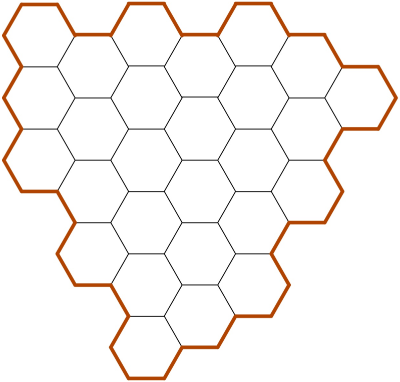
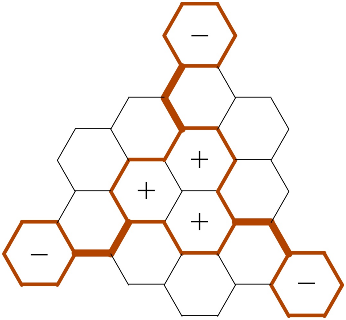




Bones tilings of the (5,7)-benzel

In both tilings, we get a signed tiling of the "signed region" given by the shadow of the boundary of the benzel:





Bones tilings are rare

For $2 \leq a, b \leq 10$ (with $a \leq 2b$ and $b \leq 2a$), the only pairs (a, b) such that the (a, b)-benzel can be tiled by bones are (5,7) and (7,5).

For all other pairs in this range, one can prove that no such tiling exists by finding a stones-and-bones tiling of the (a, b) benzel with unequal numbers of right-pointing stones and left-pointing stones, thus proving that the Conway-Lagarias invariant is nonzero.

But I don't know how to apply this approach systematically (rather than ad hoc) so that it can be used to reason about infinitely many pairs (a, b) at once.

One needs to do more than draw pictures!

Main theorem

Kim and Propp (2022): The (a, b)-benzel (can be tiled by bones if and only if there exists $k \geq 2$ such that $a = k(3k-1)/2$ and $b = k(3k+1)/2$ or vice versa.

The method of proof of the negative result (the “only if” part) is fairly algebraic. One proves that the (unrescaled) Conway-Lagarias invariant of the (a, b)-benzel is

$$\begin{array}{ll} (-3a^2+6ab-3b^2+a+b)/2 & \text{if } a+b \equiv 0 \pmod 3, \\ (a^2-4ab+b^2+a+b-2)/2 & \text{if } a+b \equiv 1 \pmod 3, \\ (-3a^2+6ab-3b^2-a-b+2)/2 & \text{if } a+b \equiv 2 \pmod 3 \end{array}$$

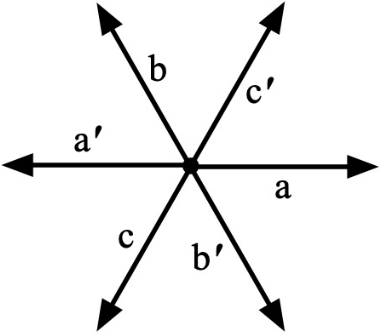
and then applies elementary number theory to show that (a, b) must be of the required form.

Describe the boundary

Let's focus on the case $a + b \equiv 0 \pmod 3$. The shadow of the boundary of the (a, b)-benzel is given by

$$(\mathrm{ab'cb'})^s(\mathrm{ba'bc'})^t(\mathrm{bc'ac'})^s(\mathrm{cb'ca'})^t(\mathrm{ca'ba'})^s(\mathrm{ac'ab'})^t \tag{1}$$

where $s = (2a - b)/3$ and $t = (2b - a)/3$ and the six unit vectors $\mathrm{a}, \mathrm{a'}, \mathrm{b}, \mathrm{b'}, \mathrm{c}, \mathrm{c'}$ are as shown:



A Gauss formula

If vectors $\mathrm{v}_1, \ldots, \mathrm{v}_n$ satisfying $\mathrm{v}_1 + \cdots + \mathrm{v}_n = 0$ are laid head-to-tail, forming a closed path, the algebraic area enclosed by the path is

$$\frac{1}{2} \sum_{1 \le i < j \le n} v_i \times v_j \tag{2}$$

where $(x_1, y_1) \times (x_2, y_2) = x_1 y_2 - y_1 x_2$.

E.g., a unit square encircled counterclockwise by a path consisting of the vectors $+i$, $+j$, $-i$, and $-j$ has area

$$\begin{aligned} &\frac{1}{2}\Big(\quad (+i)\times(+j) \; + \; (+i)\times(-i) \; + \; (+i)\times(-j) \\ &\quad + (+j)\times(-i) \; + \; (+j)\times(-j) \; + \; (-i)\times(-j) \quad \Big) \\ &= \frac{1}{2}((1)+(0)+(-1)+(1)+(0)+(1)) = \frac{1}{2}(2) = 1 \end{aligned}$$

The Conway-Lagarias invariant of a benzel

Combining (1) and (2), one concludes that there exist constants $C_1, C_2, C_3, C_4, C_5, C_6$ such that the Conway-Lagarias invariant of the (a, b) benzel with $a + b \equiv 0 \pmod 3$ is given by

$$C_1 a^2 + C_2 ab + C_3 b^2 + C_4 a + C_5 b + C_6$$

Then you can solve for $C_1, C_2, C_3, C_4, C_5, C_6$ by drawing six pictures (and solving a system of linear equations).

Similar analyses work when $a + b \equiv 1$ and $a + b \equiv 2$.

Then you can hand the problem to a number-theorist and they'll tell you that the Conway-Lagarias invariant vanishes precisely when $\{a,\ b\} = \{k(3k-1)/2,\ k(3k+1)/2\}$ for some $k \geq 2$.

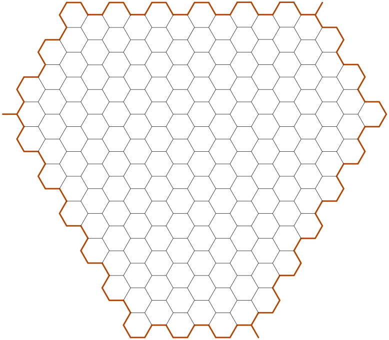
The “if”

How do we know that such a tiling exists if $\{a,\ b\} = \{k(3k-1)/2,\ k(3k+1)/2\}$ for some $k \geq 2$?

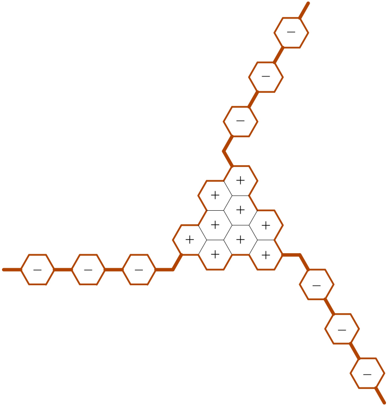
(I just used the Conway-Lagarias invariant to show you that the “only if” claim holds.)

Jesse Kim found a construction that works for all k at the Open Problems in Algebraic Conference earlier this year.

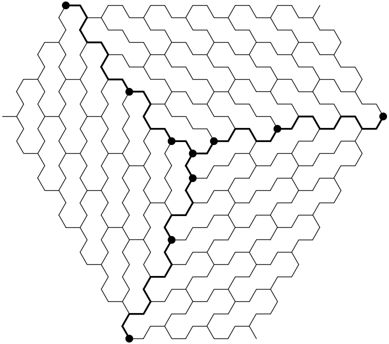
The (12,15)-benzel: the boundary



The (12,15)-benzel: the shadow of the boundary

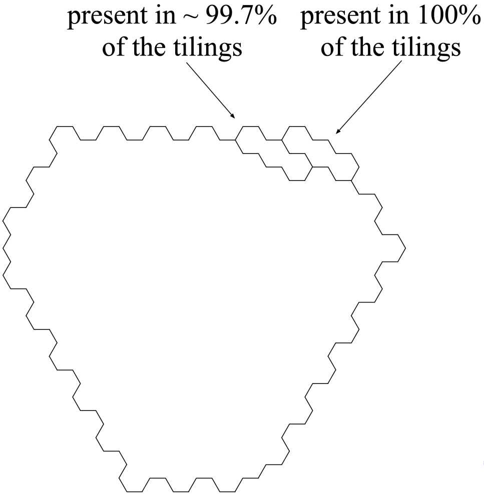


The (12,15)-benzel, tiled à la Kim



Random tilings (preview)

Thanks to David desJardins' `TilingCount` program, we see that random tribone tilings of "$k(3k \pm 1)/2$ benzels" will exhibit some sort of "freezing" near the boundary, as one sees for domino tilings of Aztec diamonds and diamond tilings of hexagons:



That's all I got

Thanks for listening!

Slides at http://jamespropp.org/msu22.pdf
Video at . . . ...