# THE FRACTIONAL CHROMATIC NUMBER OF MYCIELSKI'S GRAPHS

Michael Larsen
University of Pennsylvania
Philadelphia, PA 19104

James Propp
Massachusetts Institute of Technology
Cambridge, MA 02139

Daniel Ullman
George Washington University
Washington, DC 20052

## ABSTRACT

The most familiar construction of graphs whose clique number is much smaller than their chromatic number is due to Mycielski, who constructed a sequence $G_n$ of triangle-free graphs with $\chi(G_n) = n$. In this note, we calculate the fractional chromatic number of $G_n$ and show that this sequence of numbers satisfies the unexpected recurrence $a_{n+1} = a_n + \frac{1}{a_n}$.

## INTRODUCTION

All our graphs are finite and simple. We write $V(G)$ for the vertex set, $E(G)$ for the edge set, $\chi(G)$ for the chromatic number, and $\omega(G)$ for the clique number of $G$. We denote by $\chi_F(G)$ the fractional chromatic number (or set-chromatic number [**2**], or ultimate chromatic number [**5**], or multi-coloring number [**6**]) of $G$, which is the infimum of all fractions $a/b$ such that, to each vertex of $G$, one can assign a $b$-element subset of $\{1, 2, 3, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets.

The integer programs which compute $\chi$ and $\omega$ are dual to one another. The real relaxation of either of these integer programs is a linear program which computes $\chi_F$. Taking this viewpoint, one sees that the infimum in the definition of $\chi_F(G)$ is always achieved, that $\chi_F(G)$ is always rational, and that $\omega(G) \le \chi_F(G) \le \chi(G)$.

A fractional clique is a map $f : V(G) \to [0, 1]$ such that, if $S$ is any independent set of vertices in $V(G)$, $\sum_{v \in S} f(v) \le 1$. The fractional clique number $\omega_F(G)$ is equal to $\sup\{\sum_{v \in V(G)} f(v)\}$, where the supremum is taken over all fractional cliques $f$. This is just a combinatorial description of the parameter calculated by the real relaxation of the integer program that calculates $\omega(G)$, so $\omega_F(G) = \chi_F(G)$ by the duality theorem of linear programming.

Although there is a duality between $\omega$ and $\chi$, there is a certain lack of symmetry as well. For example, $\chi_F(G)$ is always equal to $\lim_{n\to\infty} \sqrt[n]{\chi(G^n)}$, where the power of $G$ is relative to either the disjunctive or lexicographic product of graphs (see [**5**], [**10**], [**13**]);

on the other hand, it is not true that $\omega_F(G)$ always equals $\lim_{n\to\infty} \sqrt[n]{\omega(G^n)}$. In fact, this limit gives the Shannon capacity of the complement of $G$ [**14**], which is known to be $\sqrt{5}$ when $G$ is the pentagon $C_5$ [**9**] (while $\omega_F(C_5) = \frac{5}{2}$).

## THE GRAPH TRANSFORMATION OF MYCIELSKI

Motivated by [**11**], given a graph $G$ we define a graph $\mu(G)$ as follows. If $G$ has vertex set $\{v_1, v_2, \ldots, v_m\}$, let $V(\mu(G)) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z\}$ with $x_i x_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, with $x_i y_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, with $y_i z \in E(\mu(G))$ for all $i$ from 1 to $m$, and with $\mu(G)$ having no other edges.

The proofs of parts (a) and (b) of the following theorem are implicit in [**11**] but are included here for the sake of completeness.

**Theorem.** *Suppose that $G$ has at least one edge. Then*

$$\begin{aligned} &\text{(a)}\quad \omega(\mu(G)) = \omega(G);\\ &\text{(b)}\quad \chi(\mu(G)) = \chi(G) + 1;\\ \textit{and}\quad &\text{(c)}\quad \chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}. \end{aligned}$$

**Proof**: (a) Since $G$ is an induced subgraph of $\mu(G)$, $\omega(G) \leq \omega(\mu(G))$. To see the opposite inequality, note that the vertex $z$ is in no cliques of size bigger than 2. Also, were $\{x_{i(1)}, x_{i(2)}, \ldots, x_{i(r)}, y_{j(1)}, y_{j(2)}, \ldots, y_{j(s)}\}$ a clique in $\mu(G)$, then the sets $\{i(1), \ldots, i(r)\}$ and $\{j(1), \ldots, j(s)\}$ would be disjoint and $\{v_{i(1)}, v_{i(2)}, \ldots, v_{i(r)}, v_{j(1)}, v_{j(2)}, \ldots, v_{j(s)}\}$ would be a clique. Hence $\omega(G) \geq \omega(\mu(G))$ as well.

(b) If $k : V(G) \to \{1, 2, \ldots, n\}$ is a proper coloring of $G$, then we define a coloring $h : V(\mu(G)) \to \{1, 2, \ldots, n, n+1\}$ of $\mu(G)$ by setting $h(x_i) = h(y_i) = k(v_i)$ for all $i$, and $h(z) = n + 1$. This is easily seen to be a proper coloring of $\mu(G)$ and uses only one more color than the coloring of $G$. Hence $\chi(\mu(G)) \leq \chi(G) + 1$. On the other hand, if $h$ is any proper coloring of $\mu(G)$, we define a coloring $k$ of $G$ by

$$k(v_i) = \begin{cases} h(x_i) & \text{if } h(x_i) \neq g(z); \\ h(y_i) & \text{if } h(x_i) = h(z). \end{cases}$$

This is a proper coloring of $G$ which does not use the color $h(z)$. So $G$ can be colored with fewer colors than are required to color $\mu(G)$. Hence $\chi(\mu(G)) \geq \chi(G) + 1$ as well.

(c) First, suppose $\chi_F(G) = \frac{a}{b}$ and we have a proper $a/b$-coloring of $G$. We produce an $(a^2 + b^2)/(ab)$-coloring of $\mu(G)$ as follows. Imagine that each of the $a$ colors has $a$ offspring, $b$ male and $a - b$ female. Color $x_i$ with all the offspring of the colors that are associated to $v_i$. Color $y_i$ with all the female offspring of the colors that are associated with $v_i$ and with wholly new colors $\{c_1, c_2, \ldots, c_{b^2}\}$. Color $z$ with all the male offspring of all the original colors. Note that this set coloring is proper. There are $a^2$ offspring colors and $b^2$ new colors, making $a^2+b^2$ all told. The resulting coloring of $\mu(G)$ assigns exactly $ab$ colors to each vertex. Hence $\chi_F(\mu(G)) \le \chi_F(G) + (\chi_F(G))^{-1}$.

To prove the opposite inequality, suppose $f$ is a fractional clique on $G$ that achieves $\omega_F(G)$. We define a map $g : V(\mu(G)) \to [0,1]$ as follows.

$$\begin{aligned} g(x_i) &= \left(1 - \frac{1}{\omega_F(G)}\right) f(v_i), \\ g(y_i) &= \frac{1}{\omega_F(G)} f(v_i), \\ g(z) &= \frac{1}{\omega_F(G)}. \end{aligned}$$

We now show that $g$ is a fractional clique on $\mu(G)$.

If $M \subset V(G)$, let $x(M) = \{x_i : v_i \in M\}$ and let $y(M) = \{y_i : v_i \in M\}$. Let $S$ be an independent set in $\mu(G)$. If $z \in S$, then $S = \{z\} \cup x(M)$ for some independent set $M \subset V(G)$, in which case

$$\sum_{v \in S} g(v) = \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) \le \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) = 1.$$

If $z \notin S$, then $S = x(M) \cup y(N)$ for some (independent) set $M \subset V(G)$ and some $N \subset V(G)$. Because $S$ is an independent set, $N$ may be partitioned into a subset $A$ of $M$ and a set $B$ of vertices which are neither elements of $M$ nor adjacent to elements of $M$. Then

$$\begin{aligned} \sum_{v \in S} g(v) &= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in N} f(v) \\ &= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in A} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v) \\ &\le \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v) \\ &= \sum_{v \in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in B} f(v). \qquad (*) \end{aligned}$$

Let $H$ be the subgraph of $G$ induced on $B$. Say that $\omega_F(H) = \chi_F(H) = \frac{a}{b}$ and that we have an $a/b$-coloring of $H$. Then the $a$ color classes $C_1, C_2, \ldots, C_a$ are independent sets in $H$, and the sets of the form $M \cup C_i$ are independent sets in $G$. Because $f$ is a fractional clique,

$$\sum_{v \in M} f(v) + \sum_{v \in C_i} f(v) \le 1$$

for all $i$. Adding these $a$ inequalities gives

$$a \sum_{v \in M} f(v) + b \sum_{v \in B} f(v) \leq a.$$

Dividing through by $a$ yields

$$\sum_{v \in M} f(v) + \frac{1}{\omega_F(H)} \sum_{v \in B} f(v) \leq 1.$$

Since $\omega_F(G) \geq \omega_F(H)$, $(*)$ above is less than or equal to 1, and so $g$ is a fractional clique.

It follows that

$$\begin{aligned}
\chi_F(\mu(G)) &= \omega_F(\mu(G)) \\
&\geq \sum_{v \in V(\mu(G))} g(v) \\
&= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \sum_{v \in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \omega_F(G) + \frac{1}{\omega_F(G)} \\
&= \chi_F(G) + \frac{1}{\chi_F(G)}
\end{aligned}$$

and the theorem is proved. □

Let $G_2$ be $K_2$, the complete graph on two vertices, and recursively define $G_{n+1} = \mu(G_n)$ for $n \geq 2$. This definition makes $G_3$ the 5-cycle and $G_4$ the Grötzsch graph. Our theorem shows that $G_n$ is triangle-free yet has chromatic number $n$. (This much was known to Mycielski in 1955. [**11**]) Our theorem also shows that the fractional chromatic number of $G_n$ equals $a_n$, where $a_2 = 2$ and $a_{n+1} = a_n + a_n^{-1}$. It is known that this sequence grows like $\sqrt{2n}$ in the sense that $a_n/\sqrt{2n} \to 1$ as $n \to \infty$. (See [**7**], p. 49, [**12**], problem 60, or [**1**], problem E3276 for more detailed information about the growth of this sequence.)

This provides a simple example of graphs $G_n$ with $\chi(G_n) - \chi_F(G_n) \to \infty$ and $\chi_F(G_n) - \omega(G_n) \to \infty$. In fact, even the ratios $\chi(G_n)/\chi_F(G_n)$ and $\chi_F(G_n)/\omega(G_n)$ approach infinity.

If $G$ is a graph on $v$ vertices and $\chi_F(G)$ is expressed as a fraction in lowest terms, how large can the denominator be? Dave Fisher has pointed out, as a corollary of our theorem, that $G_n$ breaks the record for the largest denominator, previously held by a sequence of graphs constructed by Chvátal, Garey, and Johnson. [**3**] The fractional chromatic number of their graphs has a denominator on the order of $e^{\sqrt{n\ln(n)/2}}$, but Fisher calculates that for $G_n$ the denominator is on the order of $e^{cn}$ for a certain constant $c$. [**4**] Fisher goes on to construct a sequence of graphs with denominators growing like $e^{cn}$ for a larger constant $c$, which gives the best known result of this type.

## ACKNOWLEDGEMENTS

Michael Larsen was supported by Graduate Fellowships from NSF and from IBM. James Propp was supported by an NSF Postdoctoral Fellowship. Daniel Ullman was partially supported by a Junior Scholar Incentive Award from George Washington University.

## References

**1**. D. M. Bloom, problem E3276, *Amer. Math. Monthly* **95** (1988), 654.

**2**. B. Bollobàs and A. Thomason, Set colourings of graphs, *Disc. Math.* **25** (1979), 27-31.

**3**. V. Chvátal, M. R. Garey, and D¿ S. Johnson, Two results concerning multicoloring, *Annals of Disc. Math.* **2** (1978), 151-154.

**4**. D. C. Fisher, Fraction colorings with large denominators, *J. Graph Theory* (to appear).

**5**. P. Hell and F. Roberts, Analogues of the Shannon capacity of a graph, *Ann. Disc. Math.* **12** (1982), 155-162.

**6**. A. J. W. Hilton, R. Rado, and S. H. Scott, A ($<5$)-colour theorem for planar graphs, *Bull. London Math. Soc.* **5** (1973), 302-306.

**7**. K. Hardy and K. S. Williams, "The Green Book: 100 Problems for Undergraduate Mathematics Competitions," Integer Press, Ottawa, 1985.

**8**. L. Lovàsz, Kneser's conjecture, chromatic number, and homotopy, *J. Combin. Theory, Ser. A* **25** (1978), 319-324.

**9**. L. Lovàsz, On the Shannon capacity of a graph, *IEEE Trans. on Info. Theory* **25** (1979), 1-7.

**10**. R. McEliece and J. Posner, Hide and seek, data storage, and entropy, *Ann. Math. Stat.* **42** (1971), 1706-1716.

**11**. J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* **3** (1955), 161-162.

**12**. D. J. Newman, *A Problem Seminar*, Springer-Verlag, New York (1982).

**13**. R. Pemantle, J. Propp, and D. Ullman, On tensor power of integer programs, *SIAM J. Disc. Math.* **5** (1992) 127-143.

**14**. C. E. Shannon, The zero-error capacity of a noisy channel, *IRE Trans. Info. Theory* **IT-2** (1956), 8-19.