Twenty Open Problems in Enumeration of Matchings

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This document is an exposition of an assortment of open problems arising from the exact enumeration of (perfect) matchings of finite graphs. Ten years ago, there were few known results of this kind, and exact enumeration of matchings would not have been recognized as a research topic in its own right. That situation began to change in the late '80s with the advent of the Aztec diamond (in work of Noam Elkies, Greg Kuperberg, Michael Larsen and myself) and with subsequent study of related problems by Bo-Yin Yang, William Jockusch, Mihai Ciucu and many others. Connections with the theory of plane partitions were brought to the fore by Greg Kuperberg, who used the matchings viewpoint to prove a previously conjectural enumeration of a symmetry class of plane partitions. Now there are many results in the field, and even more open problems, many of which are accessible to the general combinatorialist. This article discusses twenty such problems, some of which are important in their own right, and others of which are probably merely special cases of other more general (and more interesting) results. In most cases, the problem is to find or prove a general formula; in one case, it is to show that the number of matchings must be of a certain form (namely, a perfect square), and in another case, the challenge is to find a combinatorial proof of a fact whose only known proof is algebraic.

For updates on the status of these problems, see

`http://www-math.mit.edu/~propp/progress.ps.gz`.

We begin with problems related to lozenge tilings of hexagons. A **lozenge** is a rhombus of side-length 1 whose internal angles measure 60 and 120 degrees; all the hexagons we will consider will tacitly have integer side-lengths and internal angles of 120 degrees. Every such hexagon H can be dissected into unit equilateral triangles in a unique way, and one can use this dissection to define a graph G whose vertices correspond to the triangles and whose edges correspond to triangles that share an edge; this is the "finite honeycomb graph" dual to the dissection. It is easy to see that the tilings of H by lozenges are in one-to-one correspondence with the perfect matchings of G.

The a, b, c semiregular hexagon is the hexagon whose side lengths are a, b, c, a, b, c respectively. Lozenge-tilings of this region are in correspondence with plane partitions with at most a rows, at most b columns, and no part exceeding c. We may represent such hexagons by means of diagrams like

```
   AVAVAVAVA
  AVAVAVAVAVA
 AVAVAVAVAVAVA
AVAVAVAVAVAVAVA
VAVAVAVAVAVAVAV
 VAVAVAVAVAVAV
  VAVAVAVAVAV
   VAVAVAVAV
```

where A's and V's represent upward-pointing and downward-pointing triangles, respectively.

MacMahon showed that the number of such plane partitions is

$$\prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \prod_{k=0}^{c-1} \frac{i+j+k+2}{i+j+k+1}$$

(for a short, self-contained proof see section 2 of "The Shape of a Typical Boxed Plane Partition" by Henry Cohn, Michael Larsen, and myself, available at `http://www-math.mit.edu/~propp/shape.ps.gz`).

Problem 1: Show that in the $2n-1, 2n, 2n-1$ semiregular hexagon, the central location (consisting of the two innermost triangles) is covered by a lozenge in exactly one-third of the tilings.

(Equivalently: Show that if one chooses a random perfect matching of the dual graph, the probability that the central edge is contained in the matching is exactly 1/3.)

The hexagon of side-lengths $n, n+1, n, n+1, n, n+1$ cannot be tiled by lozenges at all, for in the dissection into unit triangles, the number of upward-pointing triangles differs from the number of downward-pointing triangles. However, if one removes the central triangle, one gets a region that can be tiled, and the sort of numbers one gets for small values of n are striking. Here they are, in factored form:

$$(2)$$

$$(2)\ (3)^{3}$$

$$(2)^{5}\ (3)^{3}\ (5)$$

$$(2)^{5}\ (5)^{7}$$

$$(2)^{2}\ (5)^{7}\ (7)^{5}$$

$$(2)^{8}\ (3)^{3}\ (5)\ (7)^{11}$$

$$(2)^{13}\ (3)^{9}\ (7)^{11}\ (11)$$

$$(2)^{13}\ (3)^{18}\ (7)^{5}\ (11)^{7}$$

$$(2)^{8}\ (3)^{18}\ (11)^{13}\ (13)^{5}$$

$$(2)^{2}\ (3)^{9}\ (11)^{19}\ (13)^{11}$$

$$(2)^{10}\ (3)^{3}\ (11)^{19}\ (13)^{17}\ (17)$$

```
   16     13     23     7
(2)   (11)   (13)   (17)
```

These are similar to the numbers one gets from counting lozenge-tilings of an n, n, n, n, n, n hexagon, in that the largest prime factor seems to be bounded by a linear function of n, and it ought to be possible to come up with a conjectural exact formula. What might be harder is proving it.

Problem 2: Enumerate the lozenge-tilings of the region obtained from the $n, n+1, n, n+1, n, n+1$ hexagon by removing the central triangle.

One can also look take a $2n, 2n+3, 2n, 2n+3, 2n, 2n+3$ hexagon and make it lozenge-tilable by removing a triangle from the middle of each of its three long sides, as shown below:

```
          AVAVAVAVAVAVAVAVA
         AVAVAVAVAVAVAVAVAVA
        AVAVAVAVAVAVAVAVAVAVA
       AVAVAVAVAVAVAVAVAVAVAVA
      AVAVAVAVAVAVAVAVAVAVAVAVA
      VAVAVAVAVAVAVAVAVAVAVAVAV
    AVAVAVAVAVAVAVAVAVAVAVAVAVAVA
   AVAVAVAVAVAVAVAVAVAVAVAVAVAVAVA
  AVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVA
 AVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVA
AVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVA
VAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAV
 VAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAV
  VAVAVAVAVAVAVAVAVAVAVAVAVAVAVAVAV
   VAVAVAVAVAVAVAVAVAVAVAVAVAVAVAV
    VAVAVAVAVAVAVAVAVAVAVAVAVAVAV
     VAVAVAVAVAVAVAVAVAVAVAVAVAV
      VAVAVAVAVAVAVAVAVAVAVAVAV
       VAVAVAVAVAV VAVAVAVAVAV
```

Here one obtains an equally tantalizing sequence of factorizations:

```
                 1

               7    2
             (2)  (7)

          2    4    4    2
        (2)  (7)  (11)  (13)

     10    3    8     2     4     2
   (2)   (3)  (5)  (13)  (17)  (19)

  2    2    2     3     4     4     8     4
(2)  (5)  (7)  (11)  (13)  (17)  (19)  (23)
```

Problem 3: Enumerate the lozenge-tilings of the region obtained from the $2n, 2n+3, 2n, 2n+3, 2n, 2n+3$ hexagon by removing a triangle from the middle of each of its long sides.

Let us now return to ordinary a, b, c semiregular hexagons. When $a = b = c$ ($= n$, say), there are not two but six central triangles. There are two geometrically distinct ways in which we can choose to remove an upward-pointing triangle and downward-pointing triangle from these six, according to whether the triangles are opposite or adjacent:

```
   AVAVAVA           AVAVAVA
  AVAVAVAVA         AVAVAVAVA
 AVAVA AVAVA       AVAV VAVAVA
 VAVAV VAVAV       VAVA AVAVAV
  VAVAVAVAV         VAVAVAVAV
   VAVAVAV           VAVAVAV
```

Such regions may be called "holey hexagons" of two different kinds. In the former case, the number of tilings of the holey hexagon is a nice round number (in the sense that, like the numbers tabulated above for Problems 2 and 3, it has small prime factors). In the latter case, the number of tilings is not round. Note, however, that in the latter case, the number of tilings of the holey hexagon divided by the number of tilings of the unaltered hexagon (given to us by MacMahon's formula) is equal to the probability that a random lozenge tiling of the hexagon contains a lozenge that covers these two triangles, and tends to 1/3 for large n. Following this clue, we examine the difference between the aforementioned probability (with its messy, unround numerator) and the number 1/3. The result is a fraction in which the numerator is now a nice round number. So, in both cases, we may have reason to think that there is an exact formula.

Problem 4: Determine the number of lozenge-tilings of a regular hexagon from which two of its innermost unit triangles (one upward-pointing and one downward-pointing) have been removed.

At this point, I must digress and explain how I did the exploratory work that indicated that all these numbers are "nice". Greg Kuperberg wrote a program called `dommaple`, which (with enhancements by David Wilson and myself) became a program called `vaxmaple`. One can feed `vaxmaple` an ASCII file of V's and A's like the ones shown above, and it will output Maple code which, if piped to `maple`, will output the number of tilings of the region. Moreover, as we will see below, `vaxmaple` can also count domino-tilings of regions (indeed, `dommaple` can do this too; the main difference between the two programs is that `vaxmaple` can handle lozenges as well). Then there is `vaxmacs`, which provides a way to do all this in real-time; interested readers with access to the World Wide Web can obtain copies of both programs via `http://math.wisc.edu/ propp/software.html`.

The main point I want to make here is that these programs take advantage of a result of Percus (based on work of Kasteleyn) that says that the number of matchings of a bipartite planar graph can always be calculated as the absolute value of the determinant of a modified adjacency matrix K consisting of 0's, 1's, and -1's, in which the rows correspond to vertices in one component of the bipartition and the columns correspond to vertices in the other component. In the case of lozenge tilings of hexagons and the associated matchings, it turns out that there is no need to modify signs of entries; the ordinary adjacency matrix will do.

Unpublished work of Greg Kuperberg shows that when row-reduction and column-reduction are systematically applied to the Kasteleyn matrix of an a, b, c semiregular hexagon, one can obtain the b-by-b Carlitz matrix whose i, jth entry is $\binom{a+c}{a+i-j}$. (This matrix can also be recognized as the Gessel-Viennot matrix that arises from interpreting each tiling as a family of nonintersecting lattice paths; Horst Sachs and his colleagues noticed this a number of years ago.) Such reductions do not affect the determinant, so we have a pleasing way of understanding the relationship between the Kasteleyn-Percus matrix method and the Gessel-Viennot lattice-path method. However, one can also verify that the reductions do not affect the *cokernel* of the matrix, either. On the other hand, the cokernel of the Kasteleyn matrix for the a, b, c hexagon is clearly invariant under permuting a, b, and c. This gives rise to three different Carlitz matrices that non-trivially have the same cokernel. E.g., if $c = 1$, then one gets an a-by-a matrix and a b-by-b matrix that both have the same cokernel, which can be computed by noticing that the third Carlitz matrix of the trio is just a 1-by-1 matrix whose sole entry is (plus or minus) a binomial coefficient. In this special case, the cokernel is just a cyclic group.

Greg Kuperberg poses the challenge:

Problem 5: Determine the cokernel of the Carlitz matrices, or equivalently of the Kasteleyn matrices of a, b, c hexagons, and if possible find a way to interpret the cokernel in terms of the tilings.

He points out that in the case $a = b = c = 2$, one gets a non-cyclic group as the cokernel.

Digressing for a moment from the topic of lozenge tilings, I point out that in general, the Kasteleyn matrix K is not canonically defined, in the sense that there may be many ways of modifying the signs of certain entries of the bipartite adjacency matrix of a graph so that all non-zero contributions to the determinant have the same sign. Thus, one should not expect the eigenvalues of K to possess combinatorial significance. However, the spectrum of K times its adjoint K^* is independent of which Kasteleyn matrix K one chooses (independently shown by David Wilson and Horst Sachs). Thus, it is natural to ask:

Problem 6: What is the significance of the spectrum of KK^*, where K is any Kasteleyn matrix associated with a bipartite planar graph?

Returning now to lozenge tilings, or equivalently, perfect matchings of finite subgraphs of the infinite honeycomb, consider the hexagon graph with $a = b = c = 2$:

```
     ___
 ___/   \___
/   \___/   \
\___/   \___/
/   \___/   \
\___/   \___/
    \___/
```

This is the graph whose 20 perfect matchings correspond to the 20 tilings of the regular hexagon of side 2 by rhombuses of side 1. If we just look at the probability of each individual horizontal edge belonging to a matching chosen uniformly at random ("edge-probabilities"), we get

```
    .7
.3      .3
    .3
.4      .4
    .3
.3      .3
    .7
```

Now let us look at this table of numbers as if it described a distribution of mass. If we assign the three columns x-coordinates -1 through 1, we find that the weighted sum of the squares of the x-coordinates is equal to $(.3+.4+.3)(-1)^2+(.7+.3+.7+.3)(0)^2+(.3+.4+.3)(1)^2$, or $(1.0)(-1)^2+(2.0)(0)^2+(1.0)(1)^2$, or 2. If we assign the seven rows y-coordinates -3 through 3, we find that the weighted sum of the squares of the y-coordinates is equal to $(.7)(-3)^2+(.6)(-2)^2+(.3)(-1)^2+(.8)(0)^2+(.3)(1)^2+(.6)(2)^2+(.7)(3)^2$, or 20. You can do a similar (but even easier) calculation yourself for the case $a=b=c=1$, to see that the "moments of inertia" of the horizontal edge-probabilities around the vertical and horizontal axes are 0 and 1, respectively. Using `vaxmaple` to study the case $a=b=c=n$ for larger values of n, I find that the moment of inertia about the vertical axis goes like

$$0, 2, 12, 40, 100, ...$$

and the moment of inertia about the horizontal axis goes like

$$1, 20, 93, 296, 725, ...$$

It is easy to show that the former numbers are given in general by the polynomial $(n^4 - n^2)/6$. The latter numbers are subtler; they are not given by a polynomial of degree 4, though it is noteworthy that the nth term is an integer divisible by n, at least for the first few values of n.

Problem 7: Find the "vertical moments of inertia" for the mass on edges arising from edge-probabilities for random matchings of the a, b, c honeycomb graph.

Now let us turn from lozenge-tiling problems to domino-tiling problems. A **domino** is a 1-by-2 or 2-by-1 rectangle. Although lozenge-tilings (in the guise of constrained plane partitions) were studied first, it was really the study of domino tilings in Aztec diamonds that gave current work on enumeration of matchings its current impetus. Here is the Aztec diamond of order 5:

```
    XX
   XXXX
  XXXXXX
 XXXXXXXX
XXXXXXXXXX
XXXXXXXXXX
 XXXXXXXX
  XXXXXX
   XXXX
    XX
```

(An X represents a 1-by-1 square.) A tiling of such a region by dominos is equivalent to a perfect matching of a certain (dual) subgraph of the infinite square grid. This grid is bipartite, and it is convenient to color its vertices alternately black and white; equivalently, it is convenient to color the 1-by-1 squares alternately black and white, so that every domino contains one 1-by-1 square of each color. Elkies, Kuperberg, Larsen, and Propp showed that the number of domino-tilings of such a region is $2^{n(n+1)/2}$ (where $2n$ is the number of rows), and Jockusch later found an exact formula for the number of tilings of regions like

```
    XX
   XXXX
  XXXXXX
 XXXXXXXX
XXXX XXXXX
XXXX XXXXX
 XXXXXXXX
  XXXXXX
   XXXX
    XX
```

in which two innermost squares of opposite color have been removed.

Now suppose you remove two squares from the middle of an Aztec diamond of order n in the following way:

```
    XX
   XXXX
  XXXXXX
 XXXX XXX
XXXXXXXXXX
XXXX XXXXX
 XXXXXXXX
  XXXXXX
   XXXX
    XX
```

(The two squares removed are a knight's-move apart, and subject to that constraint, they are as close to being in the middle as they can be. Up to symmetries of the square, there is only one way of doing this.) Then numbers of tilings one gets are as follows (for $n = 2$ through 10):

```
        (2)

           3
        (2)

         5
      (2)  (5)

        9    2
     (2)  (3)

        17
     (2)   (3)

        22    2
     (2)   (3)

     24    2
  (2)   (3)  (73)

   31    2    2
(2)   (3)  (5)  (11)

      47    2
   (2)   (3)  (5)
```

Note that only the presence of the large prime factor 73 makes one doubt that there is a general formula; the other prime factors are reassuringly small. Further data might make it clear what that 73 is doing there.

Problem 8: Count the domino tilings of an Aztec diamond from which two close-to-central squares, related by a knight's move, have been deleted.

One can also look at "Aztec rectangles" from which squares have been removed so as to restore the balance between black and white squares (a necessary condition for tilability). For instance, one can remove the central square from an a-by-b Aztec rectangle in which a and b differ by 1, with the larger of a, b odd:

```
   XX
  XXXX
 XXXXXX
XXXXXXXX
XXXX XXXX
 XXXXXXXX
  XXXXXX
   XXXX
    XX
```

Problem 9: Find a formula for the number of domino tilings of a $2n$-by-$(2n+1)$ Aztec rectangle with its central square removed.

What about $(2n-1)$-by-$2n$ rectangles? For these regions, removing the central square does not make the region tilable. However, if one removes any one of the four squares adjacent to the middle square, one obtains a region that is tilable, and moreover, for this region the number of tilings appears to be a nice round number.

Problem 10: Find a formula for the number of domino tilings of a $(2n-1)$-by-$2n$ Aztec rectangle with a square adjoining the central square removed.

At this point, readers who are unfamiliar with the literature may be wondering why m-by-n rectangles haven't come into the story. Indeed, one of the surprising facts of life in the study of enumeration of matchings is that Aztec diamonds and their kin have (so far) been much more fertile ground for exact combinatorics that the seemingly more natural rectangles. (For instance, Kasteleyn's exact formula for the number of domino tilings of a rectangle expresses the answer as a double product of trigonometric expressions.) There are, however, a few cases I know of in which something rather nice turns up. One is the problem of Ira Gessel that appears as Problem 20 in this document. Another is the work done by Jockusch and, later, Ciucu on why the number of domino tilings of the square is always either a perfect square or twice a perfect square. In the spirit of the work of Jockusch and Ciucu, I offer here a problem based on Lior Pachter's observation that the region

```
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXX  XXXXXXX
XXXXXX  XXXXXXXX
XXXXX  XXXXXXXXX
XXXX  XXXXXXXXXX
XXX  XXXXXXXXXXX
XX  XXXXXXXXXXXX
X  XXXXXXXXXXXXX
  XXXXXXXXXXXXXX
```

(8 dominos removed from a 16x16 square) has exactly one tiling. What if we make the intrusion half as long, as in the following picture?

```
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXX
XXX  XXXXXXXXXXX
XX  XXXXXXXXXXXX
X  XXXXXXXXXXXXX
  XXXXXXXXXXXXXX
```

That is, we take a $2n$-by-$2n$ square (with n even) and remove $n/2$ dominos from it, in a partial zig-zag pattern that starts from the corner. Here are the numbers we get, in factored form, for $n = 2, 4, 6, 8, 10$:

$$(2)\,(3)^2$$

$$(2)^2\,(3)^6\,(13)^2$$

$$(2)^3\,(3)^2\,(5)^4\,(7)^2\,(3187)^2$$

$$(2)^4\,(11771899)^2\,(27487)^2$$

$$(2)^5\,(2534588575976069659)^2$$

The factors are ugly, but the exponents are nice: we get $2^{n/2}$ times an odd square.

Perhaps this is a special case of a two-parameter fact that says that you can take an intrusion of length m in a $2n$-by-$2n$ square and the number of tilings of the resulting region will always be a square or twice a square.

Problem 11: What is going on with "intruded Aztec diamonds"? In particular, why is the number of tilings so square-ish?

Let's now get back to those Kasteleyn matrices we discussed earlier. Work of Rick Kenyon and David Wilson has shown that the *inverses* of these matrices are loaded with combinatorial information, so it would be nice to get our hands on them. Unfortunately, there are a lot of non-zero entries in the inverse-matrices. (Recall that the Kasteleyn matrices themselves, being nothing more than adjacency matrices in which some of the 1's have been strategically replaced by -1's, are sparse; their inverses, however, tend to have most if not all of their entries non-zero.) However, some exploratory numerology leaves room for hope that this is do-able.

Consider the Kasteleyn matrix K_n for the Aztec diamond of order n, in which every other vertical domino has its sign flipped (that is, the corresponding 1's in the bipartite adjacency matrix are replaced by -1's).

Problem 12: Show that the sum of the entries of the matrix inverse of K_n is $(n-1)(n+3)/2 - 2^{n-1} + 2$.

(This formula works for $n = 1$ through $n = 8$.)

I should mention in this connection that Greg Kuperberg has some high-tech ruminations on the inverses of Kasteleyn matrices, and there is a chance that representation-theory methods will be useful here.

Now we turn to a class of regions I call "pillows" on account of their agreeably lumpy shape. Here is a "0 mod 4" pillow of "order 5":

```
            XXXX
         XXXXXXXX
      XXXXXXXXXXXX
   XXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXXXXXX
 XXXXXXXXXXXXXXXX
  XXXXXXXXXXXX
   XXXXXXXX
    XXXX
```

And here is a "2 mod 4" pillow of "order 7":

```
                  XX
               XXXXXX
            XXXXXXXXXX
         XXXXXXXXXXXXXX
      XXXXXXXXXXXXXXXXXX
   XXXXXXXXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXXXXXXXXXXXX
XXXXXXXXXXXXXXXXXXXXXXXXXX
 XXXXXXXXXXXXXXXXXXXXXX
  XXXXXXXXXXXXXXXXXX
   XXXXXXXXXXXXXX
    XXXXXXXXXX
     XXXXXX
      XX
```

It turns out (empirically) that the number of 0-mod-4 pillows of order n is a perfect square times the coefficient of x^n in the Taylor expansion of $(5+3x+x^2-x^3)/(1-2x-2x^2-2x^3+x^4)$. Similarly, it appears that the number of 2-mod-4 pillows of order n is a perfect square times the coefficient of x^n in the Taylor expansion of $(5+6x+3x^2-2x^3)/(1-2x-2x^2-2x^3+x^4)$. (If you're wondering about "odd pillows", I should mention that there's a nice formula for the number of tilings, but it isn't interesting, because an odd pillow splits up into many small non-communicating sub-regions such that a tiling of the whole region corresponds to a choice of tiling on each of the sub-regions.)

Problem 13: Find a general formula for the number of domino tilings of even pillows.

Jockusch looked at Aztec diamonds with a 2-by-2 single hole in the center, and counted the number of tilings. One way to generalize this is to make the hole larger. Here's what David Wilson has to report on this:

```
We all know the formula for the number of tilings that the
Aztec diamond has.  Doug Zare asked how many tilings there
are in an Aztec diamond with an Aztec diamond deleted from
it.  Let us define the Aztec window with outer order y and
inner order x to be the Aztec diamond of order y with an
```

Aztec diamond of order x deleted from its center. For example, this is the Aztec window with orders 8 and 2:

```
       XX
      XXXX
     XXXXXX
    XXXXXXXX
   XXXXXXXXXX
  XXXXXXXXXXXX
 XXXXXX  XXXXXX
XXXXXX    XXXXXX
XXXXXX    XXXXXX
 XXXXXX  XXXXXX
  XXXXXXXXXXXX
   XXXXXXXXXX
    XXXXXXXX
     XXXXXX
      XXXX
       XX
```

There are a number of interesting patterns that show up when we count tilings of Aztec windows. For one thing, if w is a fixed even number, and y = x+w, then for any w the number of tilings appears to be a polynomial in x. (When w is odd, and x is large enough, there are no tilings.) For w=6, the polynomial is

$$8192\, x^8 + 98304\, x^7 + 573440\, x^6 + 2064384\, x^5 + 4988928\, x^4$$

$$+ 8257536\, x^3 \quad 9175040\, x^2 + 6291456\, x + 2097152.$$

Substituting x=2, the above region has 314703872 tilings. This isn't just some random polynomial. It can be rewritten as

(2)^17*x^4   1/2*x+7/8   (x+3/2)^2

where these three polynomials get composed.
For the above example, we evaluate (2+3/2)^2 = 49/4,
1/2*49/4 + 7/8 = 7, 2^(17)*7^4 = 314703872 tilings.
For all integer values of x, by the time the second polynomial is applied, the result is an integer.

Is this decomposition of the 6th Aztec window polynomial a fluke? Of course not! In general the rightmost polynomial is (x+w/4)^2, and the leftmost polynomial is either a perfect square, twice a fourth power, or half a fourth power, depending on w mod 8. A pattern for the middle polynomial however eludes me.

We all know that the Aztec window polynomials will be squareish because of symmetry, but why would they be quarticish half the time, but only perfect squares the other half of the time?
The rightmost polynomial in the decomposition is equivalent to saying that when the polynomials are expressed in terms of (3*x+y)/4 rather than x, there are no odd degree terms.
I gave these polynomials to a computer-algebra person who said that he could find functional decompositions of polynomials, but it turns out that no-one implemented his algorithm, so I don't know whether or not the polynomials decompose further.

Using the old version of vaxmacs I was able to determine the polynomials for values of w up to 14, with the new version (which is now installed at MSRI) it was not too hard to compute the polynomials for all w up to 34.

Does anybody see a pattern to these polynomials? Or how to prove the above observations? The first few polynomials are given below, and are normalized so that the left polynomial has the largest constant factor consistent with the composition of the middle and right polynomials being integer-valued for integer

```
x.

(2)^3*x^4    1    (x+1/2)^2             [w=2]

(2)^8*x^2    x+1    (x+1)^2             [w=4]

(2)^17*x^4    1/2*x+7/8    (x+3/2)^2   [w=6]

(2)^28*x^2    1/144*x^4+7/72*x^3+41/144*x^2+11/18*x+1    (x+2)^2

(2)^43*x^4    1/144*x^3+61/576*x^2+451/2304*x+967/1024    (x+5/2)^2
```

(David e-mailed much more data, but I'll omit details here.)

Problem 14: Find a general formula for the number of domino tilings of Aztec windows.

Even an argument explaining why the number of tilings for windows of inner order x and outer order $x + w$ should be given by a polynomial in x (for each fixed w) would constitute progress!

Now we come to some problems involving tiling that fit neither the domino-tiling nor the lozenge-tiling paradigm. Here the more general picture is that we have some periodic dissection of the plane by polygons, such that an even number of polygons meet at each vertex, allowing us to color the polygons alternately black or white. We then make a clever choice of a finite region R composed of equal numbers of black and white polygons, and we look at the number of "diform" tilings of the region, where a **diform** is the union of two polygonal cells that share an edge. In the case of domino-tilings, the underlying dissection of the infinite plane is the tiling by squares, 4 around each vertex; in the case of lozenge-tilings, the underlying dissection of the infinite plane is the tiling by equilateral triangles, 6 around each vertex.

Other sorts of periodic dissections have already played a role in the theory of enumeration of matchings. For instance, there is a tiling of the plane by isosceles right triangles associated with a discrete reflection group in the plane; in this case, the right choice of R gives us a region that can be tiled in 5^{n^2} ways. Similarly, in the tiling of the plane by triangles that comes from a 30 degree, 60 degree, 90 degree right triangle by repeatedly reflecting it in its edges gives rise to a tiling problem in which powers of 13 occur. I cannot include the pictures here, but I will say that one key feature of these regions R is revealed by looking at the colors of those polygons in the dissection that share an edge with the border of R. One sees that the border splits up into four long stretches such that along each stretch, all the polygons that touch the border have the same color.

One case that has not yet been settled is the case that arises from a rather symmetric dissection of the plane into equilateral triangles, squares, and regular hexagons, with 4 polygons meeting at each vertex. Empirically, one finds that the number of diform tilings is $2^{n(n+1)}$.

Problem 15: Prove that for the Aztec-type regions in the dissection of the plane into triangles, squares, and hexagons, the number of tilings of the region of order n is $2^{n(n+1)}$.

(See `http://www-math.mit.edu/~propp/dragon.ps` for a picture of one of these tilings.)

One way to get a new dissection of the plane from an old one is to refine it. For instance, starting from the dissection of the plane into squares, one can draw in every kth southwest-to-northeast diagonal. When k is 1, this is just a distortion of the dissection of the plane into equilateral triangles. When k is 2, this is a dissection that leads to finite regions for which the number of diform tilings is a known power of 2 (thanks to a theorem of Chris Douglas). But what about $k = 3$ and higher?

For instance, we have the roughly hexagonal region shown at the top of the next page (a union of square and triangular pieces, with certain boundary vertices marked with a "." so as to bring out the large-scale 2,3,2,2,3,2 hexagonal structure more clearly); it has 17920 tilings, where 17920 is $2^9 \cdot 5 \cdot 7$. More generally, if one takes an a, b, c quasi-hexagon, one finds that one gets a large power of 2 times a product of powers of odd primes in which all the primes are fairly small (and their exponents are too).

Problem 16: Find a formula for the number of diform tilings in the a, b, c quasi-hexagon in the dissection of the plane that arises from slicing the dissection into squares along every third upward-sloping diagonal.

I should mention that one reason for my special interest in Problem 16 is that it seems to be a genuine hybrid of domino tilings of Aztec diamonds and lozenge tilings of hexagons. I should also mention that I think the problem will yield to some generalization of the method of "urban renewal" (or graphical substitution) that has already been of such great use in enumeration of matchings of graphs.

```
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```

I will not go into great detail here on urban renewal (the interested reader can examine `http://www-math.mit.edu/~propp/ fpsac96.ps.gz`). Here is an overview of urban renewal: One studies not graphs but weighted graphs, with weights assigned to edges, and one does weighted enumeration of perfect matchings, where the weight of a matching is the product of the weights of the constituent edges. One then looks at local substitutions with a graph that preserve the sum of the weights of the matchings, or more generally, multiply the sum of the weights of the matchings by some predictable factor. Then the problem of weight-enumerating matchings of one graph reduces to the problem of weight-enumerating matchings of another (hopefully simpler) graph. Iterating this procedure, one can often eventually reduce the graph to something one already understands. I am confident that this will apply to Problem 16, yielding a reduction to the standard a, b, c semiregular hexagon.

Problems 15 and 16 are just two instances of a broad class of problems arising from periodic graphs in the plane. A unified understanding of this class of problems has begun to emerge, by way of urban renewal. The most important open problem connected with this class of results is the following:

Problem 17: Characterize those local substitutions that have a predictable effect on the weighted sum of matchings of a graph.

The most useful local substitution so far has been

```
        q                            q
        |                      C   /   \    D
    a   u   b                    /       \
      /   \                    /           \
p---t       v---r    ---->   p               r
      \   /                    \           /
    d   w   c                    \       /
        |                      B   \   /   A
        s                            s
```

(where unmarked edges have weight 1 and where A, B, C, D are obtained from a, b, c, d by dividing by $ac + bd$), but Rick Kenyon's substitution

```
                 3/2
p---q           p---q
|   |           |   | 1/2
r---s   ---->   r---s
|   |         2 |   |
t---u           t   u
```

has also been of use (where vertices p, q, t, u may be attached elsewhere in the graph, but not the vertices r, s).

Up till now we have been dealing exclusively with bipartite planar graphs. However, it is possible that there exists rich combinatorics involving other sorts of graphs.

For instance, one can look at the triangle graph of order n:

```
     /\
    /__\
   /\  /\
  /__\/__\
 /\  /\  /\
/__\/__\/__\
```

This particular graph has 6 matchings; we write $M(4) = 6$. More generally, we let $M(n)$ denote the number of matchings of the triangular graph whose longest row contains 6 vertices. When n is 1 or 2 mod 4, the graph has an odd number of vertices and $M(n)$ is 0; hence let us only consider the cases in which n is 0 or 3 mod 4. Here are the first few values of $M(n)$, expressed in factored form: 2, $2 \cdot 3$, $2 \cdot 2 \cdot 3 \cdot 3 \cdot 61$, $2 \cdot 2 \cdot 11 \cdot 29 \cdot 29$, $2^3 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 19 \cdot 461$, $2^3 \cdot 5^2 \cdot 37^2 \cdot 41 \cdot 139^2$, $2^4 \cdot 73 \cdot 149 \cdot 757 \cdot 33721 \cdot 523657$, $2^4 \cdot 3^8 \cdot 17 \cdot 37^2 \cdot 703459^2$, ... It is interesting that $M(n)$ seems to be divisible by $2^{\lfloor (n+1)/4 \rfloor}$ but no higher power of 2; it is also interesting that when we divide by this power of 2, in the case where n is a multiple of 4, the quotient we get, in addition to being odd, is a perfect square times a small number $(3, 11, 41, 17, \ldots)$.

Problem 18: How many perfect matchings does the triangle graph of order n have?

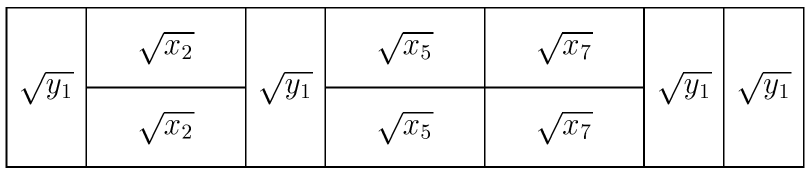
One can also look at graphs that are bipartite but not planar. A natural example is the n-cube (that is, the n-dimensional cube with all sides of length 2). It has been shown that the number of perfect matchings of the n-cube goes like 1, 2, $9 = 3^2$, $272 = 16 \cdot 17$, $589185 = 3^2 \cdot 5 \cdot 13093$, ...

Problem 19: Find a formula for the number of perfect matchings of the n-cube.

(This may be intractable; after all, the graph has exponentially many vertices.)

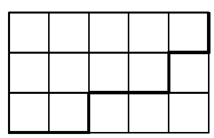
Finally, let's go to a problem involving domino tilings of rectangles, submitted by Ira Gessel (what follows are his words):

We consider dimer coverings of an $m \times n$ rectangle, with m and n even. We assign a vertical domino from row i to row $i+1$ the weight $\sqrt{y_i}$ and a horizontal domino from column j to column $j+1$ the weight $\sqrt{x_j}$. For example, the covering



for $m = 2$ and $n = 10$ has weight $y_1^2 x_2 x_5 x_7$. (The weight will always be a product of integral powers of the x_i and y_j.)

Now I'll define what I call "dimer tableaux." Take an $m/2$ by $n/2$ rectangle and split it into two parts by a path from the lower left corner to the upper right corner. For example (with $m = 6$ and $n = 10$)



Then fill in the upper left part with entries from $1, 2, \ldots, n-1$ so that for adjacent entries $\boxed{i}\boxed{j}$ we have $i < j - 1$ and for adjacent entries $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ we have $i \leq j + 1$, and fill in the lower-right partition with entries from $1, 2 \ldots, m-1$ with the reverse inequalities ($\boxed{i}\boxed{j}$ implies $i \leq j+1$ and $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}$ implies $i < j - 1$). We weight an i in the upper-left part by x_i and a j in the lower-right part by y_j.

Theorem: The sum of the weights of the $m \times n$ dimer coverings is equal to the sum of the weights of the $m/2 \times n/2$ dimer tableaux.

My proof is not very enlightening; it essentially involves showing that both of these are counted by the same formula.

Problem 20: Is there an "explanation" for this equality? In particular, is there a reasonable bijective proof? Notes:

(1) The case $m = 2$ is easy: the 2×10 dimer covering above corresponds to the 1×5 dimer tableau

x_2	x_5	x_7	y_1	y_1

(there's only one possibility!)

(2) If we set $x_i = y_i = 0$ when i is even (so that every two-by-two square of the dimer covering may be chosen independently), then the equality is equivalent to the identity

$$\prod_{i,j}(x_i + y_j) = \sum_{\lambda} s_\lambda(x) s_{\tilde{\lambda}'}(y),$$

(cf. Macdonald's *Symmetric Functions and Hall Polynomials*, p. 37.) This identity can be proved by a variant of Schensted's correspondence, so a bijective proof of the general equality would be essentially a generalization of Schensted. Several people have looked at the problem of a Schensted generalization corresponding to the case in which $y_i = 0$ when i is even.

(3) The analogous results in which m or n is odd are included in the case in which m and n are both even. For example, if we take $m = 4$ and set $y_3 = 0$, then the fourth row of a dimer covering must consist of $n/2$ horizontal dominoes, which contribute $\sqrt{x_1 x_3 \cdots x_{n-1}}$ to the weight, so we are essentially looking at dimer coverings with three rows.