A Combinatorial Interpretation for the (1,4) Sequence

Gregg Musiker

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1 Definition of the (1,4) Sequence

We let $s_1 = y$ and $s_2 = x$ and define

$$s_n = \frac{s_{n-1}+1}{s_{n-2}} \text{ for } n \text{ odd} \tag{1}$$

$$= \frac{s_{n-1}^4+1}{s_{n-2}} \text{ for } n \text{ even} \tag{2}$$

If we let $x = y = 1$, and call this sequence $s_n(1,1)$, the first few terms are:
$1, 1, 2, 17, 9, 386, 43, 8857, 206, 203321, 987, 4667522, 4729, \ldots$

Editing out the 1's and splitting this sequence into two, we get sequences:

$$a_n = 2, 9, 43, 206, 987, 4729 \tag{3}$$

$$b_n = 17, 386, 8857, 203321, 4667522 \tag{4}$$

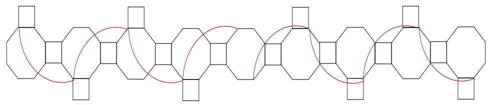
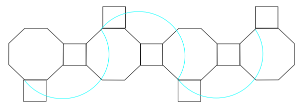
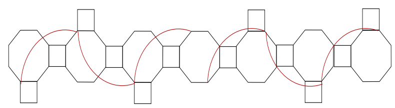
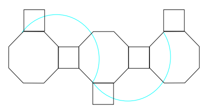
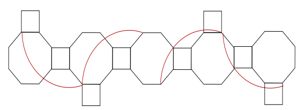
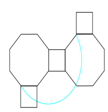
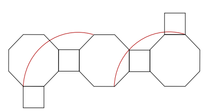
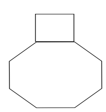
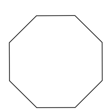
Furthermore, we can run this sequence backwards and continue the sequence:
$\ldots, 386, 9, 17, 2, 1, 1, 2, 3, 41, 14, 937, 67, 21506, 321, 493697, 1538, 11333521, 7369 \ldots$
whose negative terms split into sequences

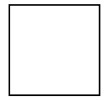
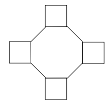
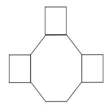
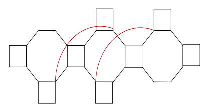
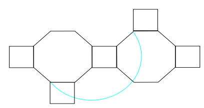
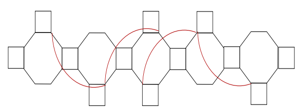
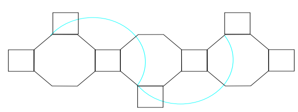
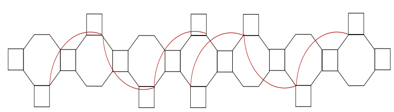
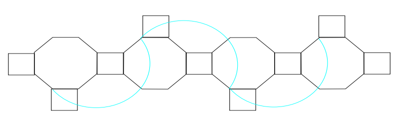
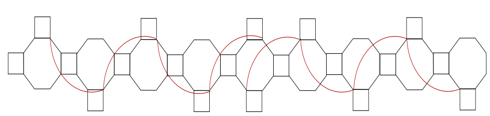
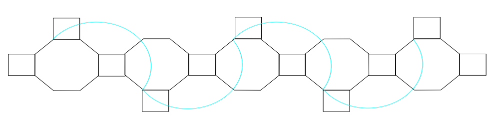
$$c_n = 2, 41, 937, 21506, 493697, 11333521 \tag{5}$$

$$d_n = 3, 14, 67, 321, 1538, 7369 \tag{6}$$

It turns out that this sequence $\{s_n(1,1)\}$ ($\{s_n(x,y)\}$) has a combinatorial interpretation as the number (weighted number) of perfect matchings in a sequence of graphs. Those graphs are given in the next two pages for several values. Notice that there are four flavors of graphs, one for each of the above four sequences (i.e. for a_n, b_n, c_n, and d_n). Notationally speaking, I will let G_n be the graph associated to s_n.

2 Graphs

n	$s_n(1,1)$	Index	Graph
-8	493697	c_5	
-7	321	d_4	
-6	21506	c_4	
-5	67	d_3	
-4	937	c_3	
-3	14	d_2	
-2	41	c_2	
-1	3	d_1	
0	2	c_1	
1	1		y
2	1		x

3	2	a_1	
4	17	b_1	
5	9	a_2	
6	386	b_2	
7	43	a_3	
8	8857	b_3	
9	206	a_4	
10	203321	b_4	
11	987	a_5	
12	4667522	b_5	
13	4729	a_6	

Notice that graph associated to a_{n+1} can be inductively built from the graph for a_n by adding two squares, one octagon, and one arc. The same is true for the sequence of d_n's. In fact, if one assumes that the graphs associated with d_n are "negative" then one can construct a_1 from d_1 by "adding" two squares and an octagon. The negative square and octagon cancels with the positive square and octagon, leaving only a square for the graph of a_1. Comparing graphs with equal numbers of octagons, there is a nice reciprocity between s_{2n+3} and s_{-2n+1}.

Similarly, b_{n+1} and c_{n+1} are constructed from b_n and c_n (resp.) by adding a complex of an octagon, two squares and an arc on both sides. Therefore there is also a reciprocity between s_{2n+2} and s_{-2n+2}. Hence the reciprocity extends to one between s_{m+2} and s_{-m+2} for all integers m.

Remark 1 *These graphs can be inductively built and satisfy a nice reciprocity, two properties reminiscent of the fibonacci numbers and its associated graphs. Additionally, the sequence of every-other fibonacci number is given by the recurrence* $f_n f_{n-2} = f_{n-1}^2 + 1$ for all n. *Perhaps any sequence of the form*

$$\begin{aligned} g_n &= \frac{g_{n-1}^a + 1}{g_{n-2}} \quad \text{for } n \text{ odd} \\ &= \frac{g_{n-1}^b + 1}{g_{n-2}} \quad \text{for } n \text{ even} \end{aligned}$$

has a combinatorial interpretation as a sequence of graphs that can be built inductively and satisfy a nice reciprocity.

Remark 2 *I also note that the cluster algebra associated with the* $(1, 4)$ *sequence can be associated to a Kac-Moody Algebra (Infinite Dimensional Lie Algebra) of affine type. The nice symmetry and periodicity of this sequence of graphs might be due to the fact that the associated Kac-Moody Algebra is affine (as opposed to hyperbolic or indefinite type).*

Remark 3 *To see some other examples of "affine" cluster algebras, look at sequences built by the recurrence* $g_n g_{n-k-1} = g_1 g_k + 1$ *as explored in my article about spines (found on the REACH website). These have rank* $k+1$ *and can also be built inductively from buliding blocks. They seem to correspond to the Kac Moody affine algebras of type* $A_l^{(1)}$.

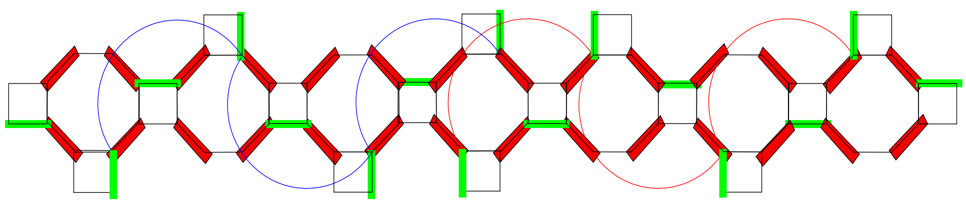
3 Weighted Version of these Graphs

One can also give these graphs weights and then $s_n(x, y)$ gives rise to a sequence of Laurent polynomials which again have a combinatorial interpretation related to these graphs. Recall that a Laurent polynomial is a rational function with a single monomial as the denominator. Given a Laurent polynomial

$$s_n(x, y) = \frac{P_n(x, y)}{x^{oct} y^{sq}}$$

and the graph G_n, $P_n(x, y)$ is the weighted number of perfect matchings in G_n and the graph G_n consists of *oct* octagons, *sq* squares and additional *arc*s. To construct these weighted graphs, we take the graphs G_n and assign weights such that each of the squares have one edge of weight x and three edges of weight 1 while the octagons have weights alternating between y and 1. Also the extra *arc*s will be given weights of x.

As an example, consider the following close-up of the graph associated to s_{10}.



The graph associated to s_{10}.

The vertical and horizontal edges colored in green are given weight x, the diagonal edges marked in red are given weight y, and the arcs are given weight x. All other edges are given weight 1.

4 Proofs

For $n \neq 1$ or 2, $s_n = P_n / x^{oct} y^{sq}$. The base cases asserting that P_{-1}, P_0, P_3, and P_4 count the number of weighted matchings in graphs G_{-1}, G_0, G_3 and G_4, respectively, can be checked by inspection. Thus the proof that P_n counts the number in G_n for all n will be proven by verifying the two recurrences. First we will prove

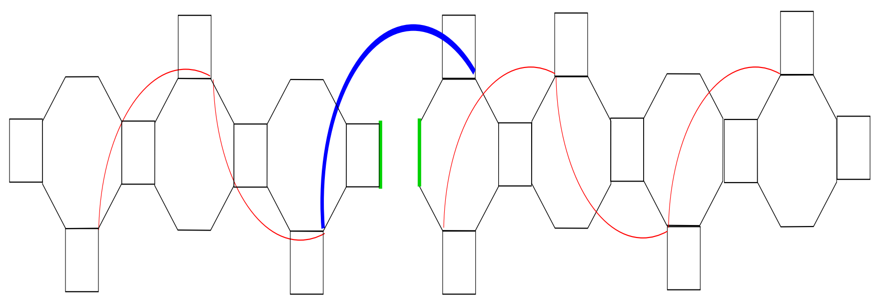
$$P_{2n} + x^{2n-3} y^{4n-4} = P_{2n-1} P_{2n+1}. \tag{7}$$

We will create two new graphs, first we will take G_{2n-1} and reflect it horizontally and call this new graph F_{2n-1}. If n is odd, we also reflect it vertically. Second, we will transform G_{2n+1} by reflecting it vertically if n is even, and then rotating an outside square for any n. We will call this new graph H_{2n+1}. These reflections and transformations will not change the number (or weighted number) of matchings. Thus P_{2n-1} will equal the weighted number of matchings in the graph F_{2n-1} or for that matter H_{2n-1}.

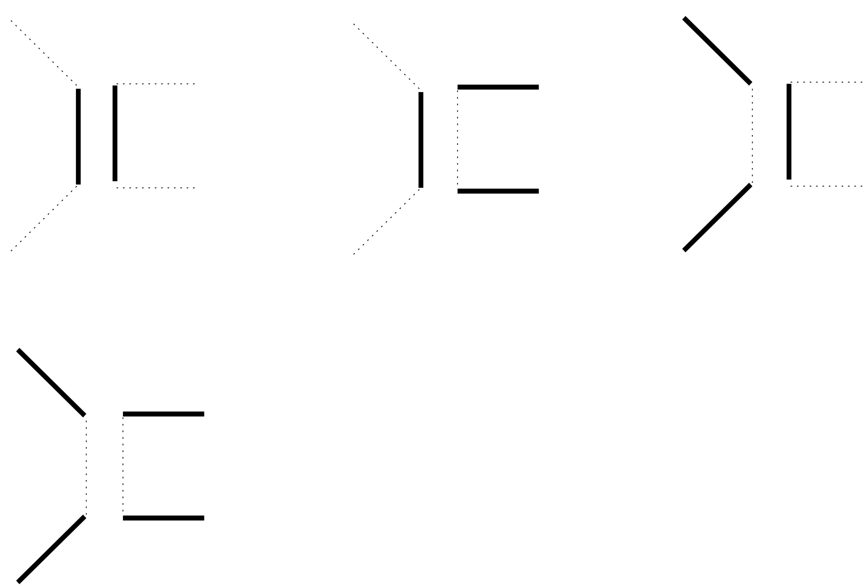
The graph G_{2n} can be decomposed

$$G_{2n} = F_{2n-1} \cup H_{2n+1} \cup \textit{middle arc}$$

where F_{2n-1} and H_{2n+1} are joined together on an overlapping edge. See picture below for example with G_{10}.



A matching of F_{2n-1} and a matching of H_{2n+1} will meet at the edge of incidence in one of the following four ways:



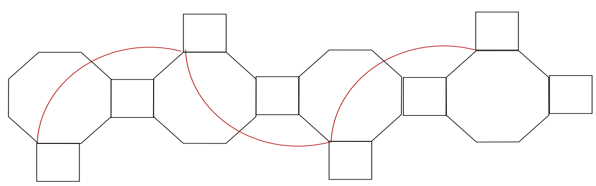
In the first three cases we can bijectively associate a matching of $F_{2n-1} \sqcup H_{2n+1}$ to a matching of G_{2n} by removing an edge of weight one on the overlap. In the last case, all the edges have nontrivial weight and cannot all appear in a matching of G_{2n} and thus we have a problem. Furthermore, we have neglected matchings of G_{2n} that utilize the *middle arc.*

More formally, we have a bijection between $\{$matchings of $(G_{2n} - \textit{middle arc})\ \}$ and the set $\{$matchings of $F_{2n-1} \sqcup H_{2n+1}\} - \{$pairs with nontrivial incidence$\}$. Thus proving $P_{2n} + x^{2n-3}y^{4n-4} = P_{2n-1}P_{2n+1}$ reduces to proving the following claim.

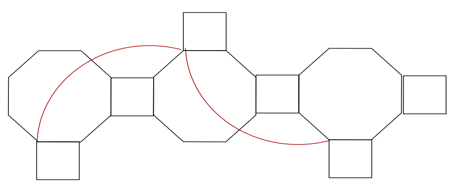
Lemma 1 *The weighted number of matchings in G_{2n} which uses middle arc is $x^{2n-3}y^{4n-4}$ less than the number of matchings in $F_{2n-1} \sqcup H_{2n+1}$ which have nontrivial incidence.*

Proof. (for $n > 1$)
Let $\tilde{P}_{2n-1}$ = the weighted number of matchings in $\tilde{G}_{2n-1}$, which is graph $G_{2n-1} -$ *outside square.*

$\tilde{G}_{11}$



$\tilde{G}_{9}$

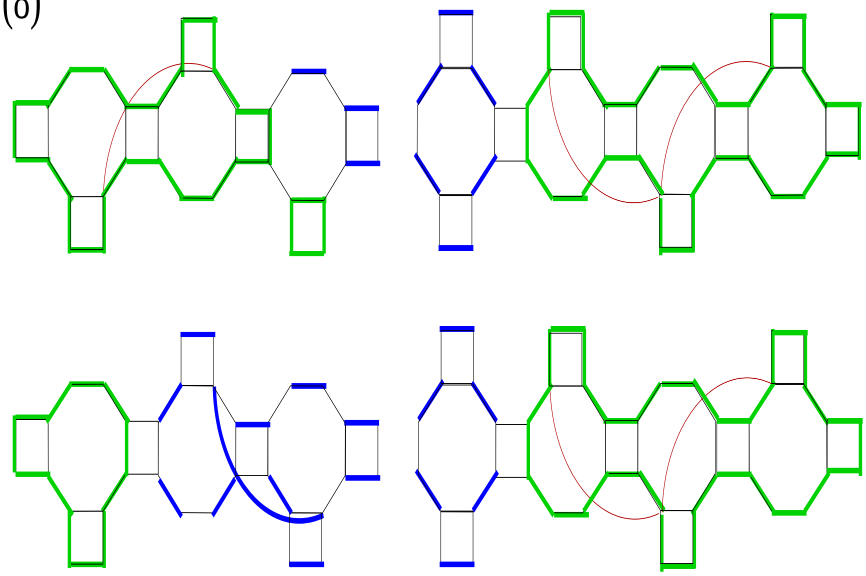


Then by analyzing (see next page) what other edges nontrivial incidence (or use of the *middle arc*) forces, we arrive at the expressions

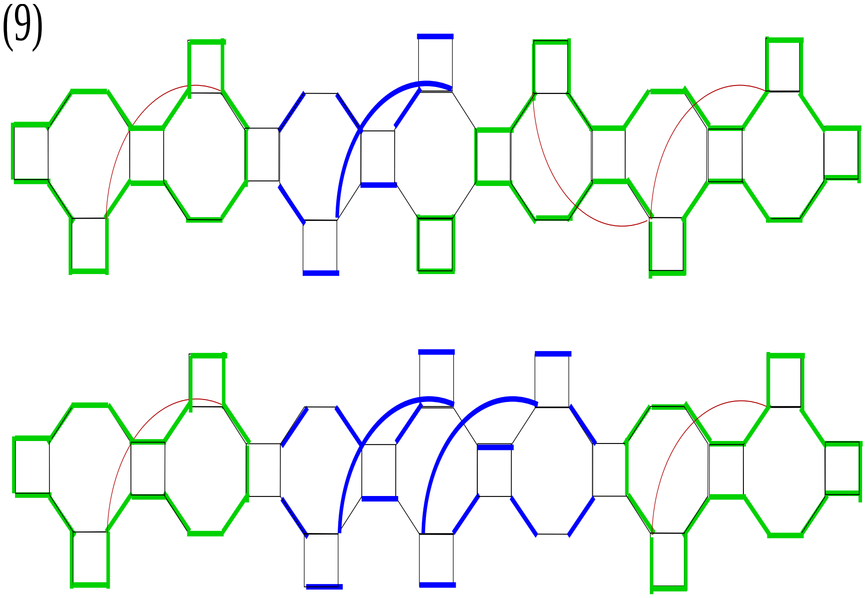
$$(x(x+1)P_{2n-3} + x^2y^4\tilde{P}_{2n-5})y^4\tilde{P}_{2n-1} \tag{8}$$
$$x(x+1)y^4\tilde{P}_{2n-3}P_{2n-1} + x^2y^8\tilde{P}^2_{2n-3} \tag{9}$$

for the weighted numbers of matchings of $G_{2n-1} \cup G_{2n+1}$ with nontrivial incidence (and the weighted number of matchings of G_{2n} using the *middle arc*), respectively.

(8)



(9)



So after the bijections in the first three cases of incidence, it suffices to prove that (8) = (9) + $x^{2n-3}y^{4n-4}$. Rearrangeing this equality, we reduce this to proving

$$x(x+1)y^4(P_{2n-3}\tilde{P}_{2n-1} - P_{2n-1}\tilde{P}_{2n-3}) = (x+1)x^{2n-4}y^{4n-4} \tag{10}$$

$$x^2y^8(\tilde{P}_{2n-3}^2 - \tilde{P}_{2n-1}\tilde{P}_{2n-5}) = x^{2n-4}y^{4n-4} \tag{11}$$

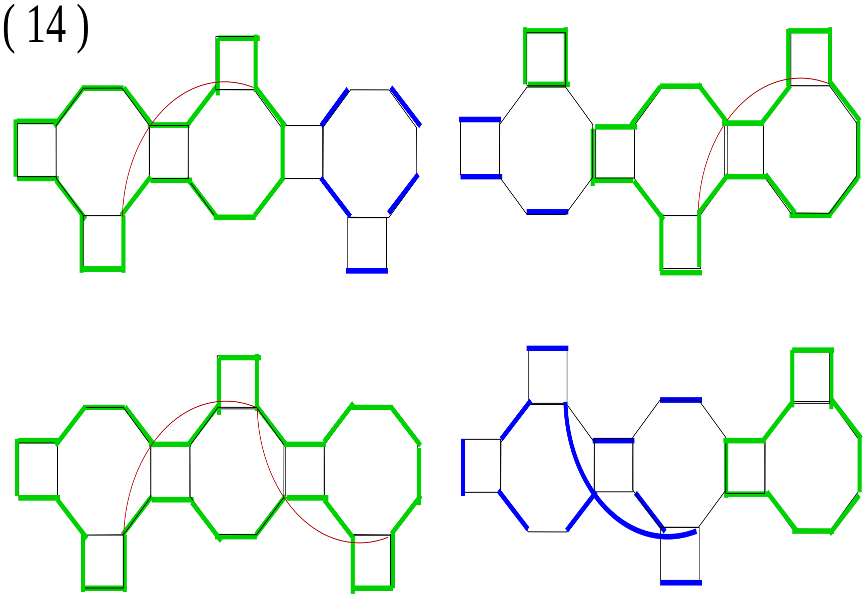
and after dividing to normalize, the equations become

$$P_{2n-3}\tilde{P}_{2n-1} - P_{2n-1}\tilde{P}_{2n-3} = x^{2n-5}y^{4n-8} \tag{12}$$

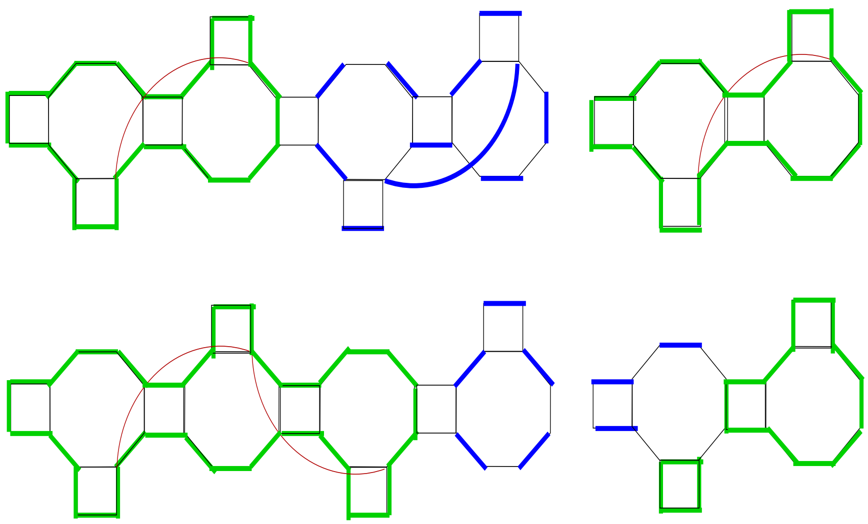
$$\tilde{P}_{2n-3}^2 - \tilde{P}_{2n-1}\tilde{P}_{2n-5} = x^{2n-6}y^{4n-12} \tag{13}$$

We prove equation (13) by making superimposed graphs involving $\tilde{P}_{2n-1}$ and $\tilde{P}_{2n-5}$ and comparing it to superimposed graphs of $\tilde{P}_{2n-3}^2$. Analogous to the analysis that allowed us to reduce from recurrence (7) to Lemma 1 by forming incidences and a bijection in the three of the case and reduce our counting to the case of nontrivial incidence or use of the *middle arc*s.

(14)



(15)



After accounting for the forced edges, we find the following two expressions:

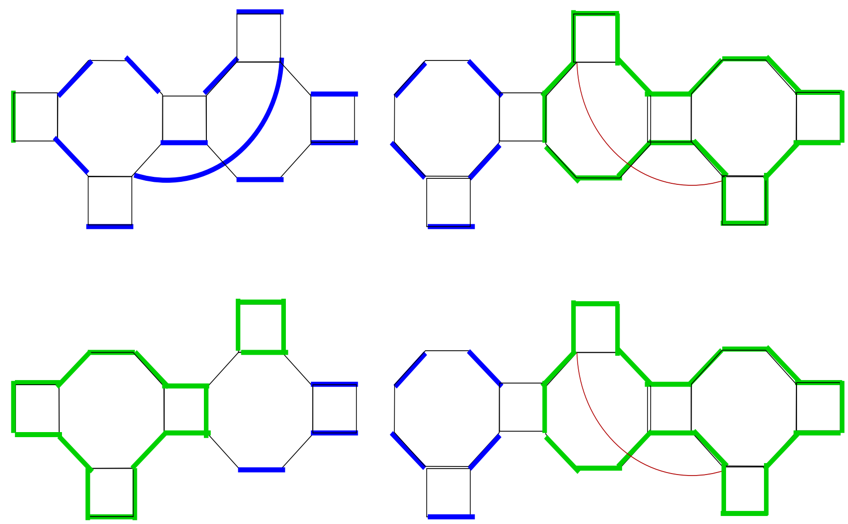
$$\tilde{P}^2_{2n-5}x(x+1)y^4 + \tilde{P}_{2n-3}\tilde{P}_{2n-7}xy^4 \quad (14)$$

$$\tilde{P}_{2n-7}\tilde{P}_{2n-3}x(x+1)y^4 + \tilde{P}^2_{2n-5}xy^4. \quad (15)$$

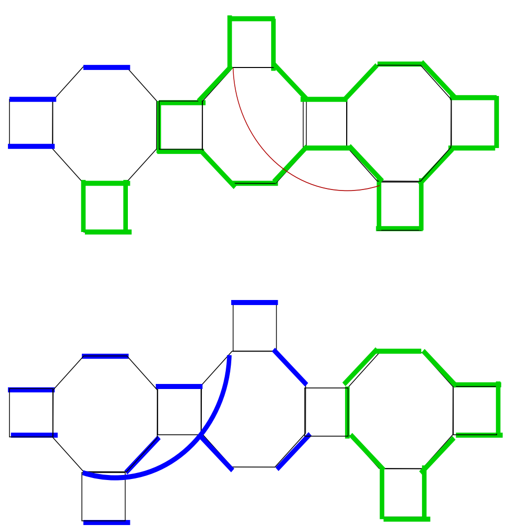
So $\tilde{P}^2_{2n-3} - \tilde{P}_{2n-1}\tilde{P}_{2n-5} = x^2y^4(\tilde{P}^2_{2n-5} - \tilde{P}_{2n-3}\tilde{P}_{2n-7})$ and so after a simple check of the base case we get equation (13) by induction.

Since we can combine P_{2n-3} and $\tilde{P}_{2n-1}$ (as well as P_{2n-1} and $\tilde{P}_{2n-3}$) into a superimposed graph, again just like earlier, we can form a bijection between these two superimposed graphs in the case of three of the incidences and reduce it to the case of nontrivial incidence.

(16)



(17)



In the case of nontrivial incidence, after highlighting the edges that are forced, the weights of the two superpositions are

$$(x(x+1)P_{2n-5} + x^2y^4\tilde{P}_{2n-7})y^4\tilde{P}_{2n-3} \quad (16)$$
$$x(x+1)y^4\tilde{P}_{2n-5}P_{2n-3} + x^2y^8\tilde{P}_{2n-5}^2 \quad (17)$$

These equations are just (8) and (9) with $(n-1)$ plugged in place of n, so by induction, their difference is $x^{2n-5}y^{4n-8}$ as required. This proves equation (12). Since the proof of equations (12) and (13) was sufficient, thus recurrence (7) is proved for $n > 1$. □

By a little tweaking, analogous analysis proves Lemma 1, hence recurrence (7) for the case $n < 1$. We define $\tilde{G}_{-2n+1}$ to be the same as $\tilde{G}_{2n+3}$ by reciprocity. (i.e. $\tilde{G}_{-2n+1}$ is graph $G_{-2n-1} \cup$ *inside square*.) Notice the similarity in definition. We will find for $n < 1$, the following two expressions in place of (8) and (9) for the weighted number of matchings of $G_{-2n-1} \cup G_{-2n+1}$ with nontrivial incidence (G_{-2n} using the *middle arc*) :

$$xy^4\tilde{P}_{-2n+1}(P_{-2n+3} + x\tilde{P}_{-2n+5}) \quad (18)$$
$$xy^4\tilde{P}_{-2n+3}(P_{-2n+1} + x\tilde{P}_{-2n+3}). \quad (19)$$

These also reduce to expressions like (12) and (13), with $-2n+k$ replacing $2n-k$, hence the recurrence is true for all $n \neq 1$.

5 Second Recurrence

Now we prove the recurrence

$$P_{2n-1}^4 + x^{4n-8}y^{8n-12} = P_{2n-2}P_{2n}. \quad (20)$$

We first mention two key observations.

Lemma 2 $P_{2n-3}P_{2n+1} = P_{2n-1}^2 + x^{2n-5}y^{4n-8}(y^4 + (x+1)^2)$.

To prove this quadratic (bilinear) recurrence, we superimpose G_{2n-3} centered on top of G_{2n+1} and compare it to a decomposition involving two copies of G_{2n-1} off-set to the left and right.

Lemma 3 $P_{2n+1} = (y^4 + (x+1)^2)P_{2n-1} - x^2y^4P_{2n-3}$.

This proof is a simple inclusion-exclusion argument relying on the fact that G_{2n+3} is inductively built from G_{2n+1} by adding an octagon, an arc, and two squares.

Using the first recurrence (7), we can rewrite the right-hand side of (20) as

$$(P_{2n-1}P_{2n-3} - x^{2n-5}y^{4n-8})(P_{2n-1}P_{2n+1} - x^{2n-3}y^{4n-4}) \tag{21}$$

which reduces to

$$P_{2n-1}^2(P_{2n+1}P_{2n-3}) - P_{2n-1}x^{2n-5}y^{4n-8}(P_{2n+1} + x^2y^4P_{2n-3}) + x^{4n-8}y^{8n-12}. \tag{22}$$

Using Lemma 2 and Lemma 3, this equation simplifes to $P_{2n-1}^4 + x^{4n-8}y^{8n-12}$. Thus the recurrence (20) is proved. □

Like before, we can extend this result to case $n < 1$ and thus recurrence (20) is proved for all $n \neq 1$. With the defining recurrences verified, it follows from the base cases that $s_n = P_n/x^{oct}y^{sq}$ where P_n is the weighted number of matchings in graph G_n.

Remark 4 *Since the numerator is the weighted number of matchings, s_n is a positive Laurent polynomial for all n.*