

# 1 Background

For a matrix  $A$ , let  $A(i, j)$  denote its  $(i, j)$ th entry. Suppose  $A_n$  is an  $n$ -by- $n$  matrix and  $A_{n+1}$  is an  $(n + 1)$ -by- $(n + 1)$  matrix. We define  $A_k$  recursively in terms of  $A_{k+1}$  and  $A_{k+2}$  by

$$A_k(i, j)A_{k+2}(i + 1, j + 1) = A_{k+1}(i, j)A_{k+1}(i + 1, j + 1) + \lambda A_{k+1}(i + 1, j)A_{k+1}(i, j + 1)$$

if  $A_{k+2}(i + 1, j + 1) \neq 0$  and  $A_k(i, j) = 0$  otherwise, in which case we write  $A_n \rightarrow A_{n-1} \rightarrow \dots \rightarrow A_1$ . The  $\lambda$ -determinant of a pair  $(A_n, A_{n+1})$  is the sole entry of  $A_1$ . Let the  $\lambda$ -determinant of a matrix  $A_n$  be the  $\lambda$ -determinant of the pair  $(A_n, C)$ , where  $C$  is the  $(n + 1)$ -by- $(n + 1)$  matrix each of whose entries is 1. Note that the  $(-1)$ -determinant of a matrix is just its determinant. Let  $\Lambda(A)$  denote the 1-determinant of  $A$ .

Let the *Aztec diamond graph* be the dual graph of an Aztec diamond, and a *weighted Aztec diamond graph (WAD)* be an Aztec diamond graph with associated edge weights. Designate by  $W(F)$  the sum of the weighted perfect matchings of a WAD  $F$ . An  $S$ -WAD is a WAD whose entries are chosen from the set  $S$ ; in this paper we will be using  $\{0, 1\}$ -WADs almost exclusively.

[DEFINE AZTEC OCTAGONS HERE]

# 2 Kuo Recurrence

Consider the Aztec diamond graph as tilted 45 degrees. A *face* of an Aztec diamond graph of order  $n$  corresponds to any bounded face of an Aztec diamond graph of order  $n + 1$ , which include unbounded faces of the Aztec diamond graph of order  $n$ . Call two faces  $F, G$  of the graph *vertex adjacent* (in which case we write  $F|G$ ) if they share a common vertex but not an edge, and *vertex connected* if either  $F|G$  or there is a finite set  $\{F_1, F_2, \dots, F_n\}$  of faces in the graph such that  $F_i|F_{i+1}$  for all  $i$ ,  $F|F_1$ , and  $F_n|G$ . Call a face a *major face* if it is vertex connected to the upper left face and a *minor face* otherwise. The major faces of an Aztec diamond  $F$  form a finite square lattice; call the one in column  $i$ , row  $j$   $F(i, j)$ . Similarly, the minor faces of  $F$  form a finite square lattice; label them  $\overline{F}(i, j)$ . If  $F$  is a WAD, let  $F(i, j)_{ne}, F(i, j)_{nw}, F(i, j)_{se}, F(i, j)_{sw}$  represent the weightings of the northeast, northwest, southeast, and southwest edges bordering  $F(i, j)$ , respectively.

Given a WAD  $F$  of order  $n$ , let  $F_{ne}, F_{nw}, F_{se}, F_{sw}$  be the northeast, northwest, southeast, and southwest order  $n - 1$  weighted Aztec subdiamonds, and let  $F_m$  be the inner order  $n - 2$  weighted Aztec subdiamond. If we let the *edge multiplying factors*  $n_e, n_w, s_e, s_w$  be the weights of the northeast, northwest, southeast, and southwest edges of  $F$ , then the number of weighted matchings of  $F$  is given in [1] by the recurrence

$$W(F)W(F_m) = n_e s_w W(F_{nw})W(F_{se}) + n_w s_e W(F_{ne})W(F_{sw}). \quad (1)$$

## 2.1 Edge Weights

Given a  $\{0, 1\}$ -WAD  $F$ , create a square matrix  $M$  whose  $(i, j)$ th entry is equal to  $F(i, j)_{nw}F(i, j)_{se} + F(i, j)_{ne}F(i, j)_{sw}$  (the *edge factor* of  $F(i, j)$ ). If  $F$  is weighted all 1 within some Aztec suboctagon and in a brickwork pattern outside this region (in which case we say that the octagon

is *embedded* in the WAD), the 1-determinant of  $M$  is equal to the number of weighted matchings of  $F$ . This comes almost immediately from (1), the only trick being that it works in general only if the possible weights are 0 and 1, in which case the edge multiplying factors drop out.

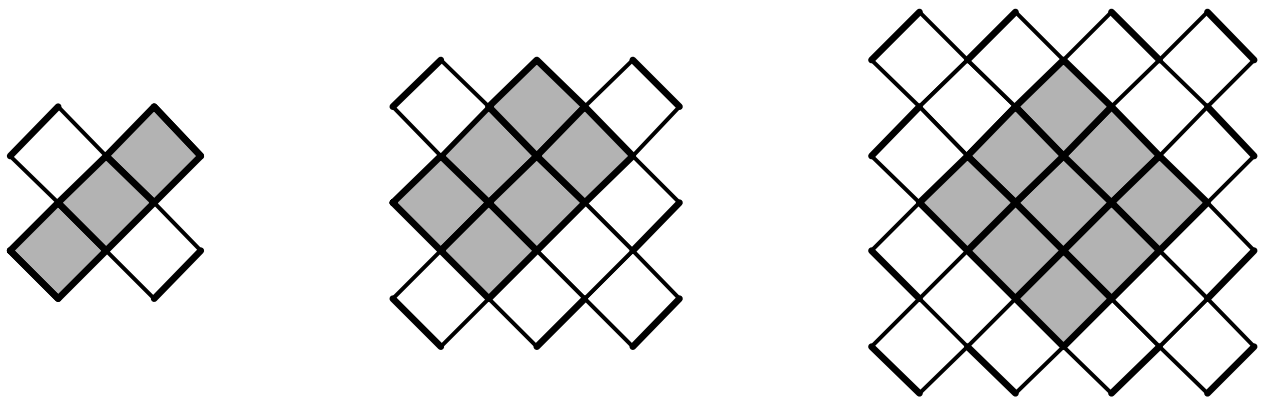
## 2.2 Face Weights

Given an order  $n$  Aztec diamond graph  $F$ , the *total minor faces* of  $F$  are the canonically-defined  $\overline{F}(i, j)$  where  $i, j$  range from 0 to  $n+1$  (i.e. defined such that they uphold the lattice structure of the minor faces). The *face weight* of a total minor face  $\overline{F}(i, j)$  is either (i) the number of edges selected immediately around  $\overline{F}(i, j)$  in a nonzero weighted perfect matching of  $F$  if this is a fixed value in any such matching, or (ii) 1 otherwise.

Given an order  $n$   $\{0, 1\}$ -WAD  $F$  whose edges are all weighted 1 within some Aztec suboctagon, and the rest of whose edges are weighted in a brickwork pattern [CLARIFY BRICKWORK PATTERN], let  $M$  be the  $(n+1)$ -by- $(n+1)$  matrix whose  $(i, j)$ th entry is the face weight of  $\overline{F}(i, j)$ . Then if  $F$  is weighted all 1 within some Aztec suboctagon and in a brickwork pattern outside this region, the 1-determinant of  $M$  is equal to the number of weighted matchings of  $F$ . The proof is immediate, since the intermediate  $n$ -by- $n$  matrix used to take the 1-determinant is equal to the matrix used in the edge weighting picture.

## 3 Embedding Rectangles

The above techniques can be used to find the number of perfect matchings of rectangle graphs, which are Aztec octagons, by embedding them in WADs via appropriate weighting. All edges of the embedded rectangle should be weighted 1, and the rest of the edges of the Aztec diamond must be weighted in a brickwork pattern (if the can be; otherwise we must choose a different embedding of the rectangle). Note that if a given rectangle cannot be embedded in any WAD, it is odd-by-odd and thus has zero matchings. In the following examples, an edge weight of 1 is represented by a darkened line, and all other edges are weighted 0. The shading is included only to highlight the region under consideration.



In this figure the leftmost example corresponds to  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow 5$ , the middle example

to  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 4 \\ 4 & 1 \end{bmatrix} \rightarrow 11$ , and the rightmost example to  $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow$

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & 8 & 4 \\ 1 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix} \rightarrow 36.$$

[THIS PICTURE WILL BE MADE TO LOOK MUCH NICER, AND MORE WILL BE ADDED, INCLUDING ONES WITH FACE AND EDGE WEIGHTINGS RIGHT ON THE PICTURE. AND THE SPACING ISSUE WILL BE FIXED. HOW MANY EXAMPLES ARE GOOD TO ADD?]

## References

- [1] E. Kuo, *Applications of Graphical Condensation for Enumerating Matchings and Tilings*, preprint 2002