The real story about .999...

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These slides are on the web at

http://jamespropp.org/real-story.pdf

The video of this talk is on the web at

http://jamespropp.org/real-story.mp4

Fundamental question: What does an expression like ".999..." mean?

If it's supposed to mean "The number you eventually reach if you write enough 9's after the decimal point", then it doesn't exist!

If it's supposed to mean "The number you **approach** (but never reach) if you keep writing 9's", then it equals 1, but you have to explain what "approach" means, and explain why the numbers .9, .99, .999, ... approach 1.

A full understanding involves calculus.

A proof you commonly see:

Let $\qquad x = 0.999\ldots$

Then $10x = 9.999\ldots$

So $10x - x = 9$.

So $9x = 9$, and $x = 1$.

Who's seen this before?

Variant: Let $x = 9/10 + 9/100 + 9/1000 + \ldots$
(an infinite sum, also known as an infinite series).
Then

$$\begin{aligned} 10x &= 90/10 + 90/100 + 90/1000 + \ldots \\ &= 9/1 + 9/10 + 9/100 + \ldots \\ &= 9 + x. \end{aligned}$$

So $9x = 9$ and $x = 1$.

What do you think?

Compare with this:

Let $x = 1 - 1 + 1 - 1 + \dots$.

Then

$$\begin{aligned} 1 - x &= 1 - (1 - 1 + 1 - 1 + \dots) \\ &= 1 - 1 + 1 - 1 + \dots \\ &= x. \end{aligned}$$

So $1 - x = x$, so $x = 1/2$.

What do you think?

Or:

Let $x = 1 + 10 + 100 + 1000 + \dots$

Then $10x = 10 + 100 + 1000 + \dots$

So $x - 10x = 1$.

So $-9x = 1$, and $x = -1/9$.

What do you think?

Or:

Let $x = 1/2 - 1/3 + 1/4 - 1/5 + 1/6 - 1/7 + 1/8 - 1/9 + \dots$

x is positive because we can write it as
$(1/2 - 1/3) + (1/4 - 1/5) + \dots$, a sum of positive numbers,

and x is negative because we can write it as
$(1/2-1/3-1/5)+(1/4-1/7-1/9)+(1/6-1/11-1/13)+\dots$,
a sum of negative numbers.

Moral: Infinite sums are tricky!

Is there a way to understanding infinite decimals without using infinite sums?

Yes: using nested intervals on the number line!

(Not a new idea, but not broadly appreciated at the pre-college level.)

The $.999\ldots$ problem is less than 400 years old.

The modern decimal system for representing real numbers was perfected by Simon Stevin around 1634.

But mathematicians were studying real numbers like $\sqrt{2}$ and π long before that.

So if the mathematician Archimedes didn't have infinite decimals, what did he mean when he made claims like "π is bigger than 3 but less than 22/7"?

Greeks didn't have real numbers.

They had counting numbers, magnitudes (weights, angles, lengths, areas, etc.), and **ratios** (which correspond to what we call positive real numbers).

They couldn't divide the circumference (C) of a circle by the diameter (D) of that circle.

They couldn't say that $C/D \approx 3$, because C and D weren't numbers; they were magnitudes.

But they could say that $C \approx 3D$ (where $3D$ means $D + D + D$).

More precisely, they could say that $C > 3D$.

Likewise, instead of saying $C/D < 22/7$, they could say that $7C < 22D$.

Notice that the lengths C and D are comparable in the sense that there's a multiple of C that's bigger than D and there's a multiple of D that's bigger than C.

Archimedean Property: If K and L are lengths, there must exist an m such that $mK > L$ and an n such that $nL > K$.

Compare: $K =$ length of mouse, $L =$ length of elephant.

Translating things into modern terms, Archimedes was saying that you can't have a ratio of lengths that's less than $1/1, 1/2, 1/3, 1/4, \ldots$.

"Infinitesimal ratios" don't exist.

Likewise you can't have a ratio of lengths that's bigger than $2/1, 3/1, 4/1, 5/1, \ldots$.

"Infinite ratios" don't exist.

The modern counterpart of the ratios of Greek mathematics are the positive real numbers.

So the Archimedean Property can be stated as: "There is no real number that is bigger than every positive integer, and there is no positive real number that is smaller than the reciprocal of every positive integer."

This may seem obvious if you think of real numbers as infinite decimals.

But in the Greek picture, this is thinking about things backward.

If you want to write a ratio as a finite or infinite decimal, you have to start by writing down its integer part.

The Archimedean Property is what guarantees that the integer part exists!

Before we apply the Archimedean Property to $.999\ldots$, let's get familiar with ratios in the Greek manner.

Pi is hard to work with, so let's use the square root of 2 instead.

Make a not-too-big square out of paper.

Let s be the length of the side of your square, and let d be the length of the diagonal.

Which is bigger: $3s$ or $2d$?

Which is bigger: $4s$ or $3d$?

Which is bigger: $7s$ or $5d$?

That last one was close, but we can compare by squaring (and using the Pythagorean Theorem):

$(7s)^2 = 49s^2$ while $(5d)^2 = 25d^2 = 25(s^2 + s^2) = 50s^2$.

So $7s$ is a hair smaller than $5d$.

That is, $7/5$ is a hair smaller than $d/s = \sqrt{2}$.

(Check this using decimals: $7/5 = 1.4 < 1.414... = \sqrt{2}$.)

The Greeks found some great approximations to $\sqrt{2}$:

$$\frac{1}{1} < \frac{3}{2} > \frac{7}{5} < \frac{17}{12} > \frac{41}{29} < \frac{99}{70} > \dots \text{ (going on forever)}$$

What's the pattern?

$$7 + 5 = 12$$

$$5 + 12 = 17$$

$$17 + 12 = 29$$

$$12 + 29 = 41$$

$$41 + 29 = 70$$

$$29 + 70 = 99$$

etc.

In modern terms, we'd say that

$$1/1 < \sqrt{2} < 3/2,$$

$$7/5 < \sqrt{2} < 17/12,$$

$$41/29 < \sqrt{2} < 99/70,$$

etc.: an infinite list of inequalities.

These intervals get narrower and narrower, and $\sqrt{2}$ is the only number that lies in all of them (try to prove this!).

Let's use this approach to study the infinite decimal $x = .3333\ldots$.

$$.3 < x < .4,$$

$$.33 < x < .34,$$

$$.333 < x < .334,$$

etc.

It's not hard to show that $1/3$ satisfies all these inequalities. But do these inequalities uniquely specify $1/3$?

Could there be more than one number satisfying all those inequalities?

Suppose there were two numbers x and y satisfying

$$.3 < x, y < .4,$$

$$.33 < x, y < .34,$$

$$.333 < x, y < .334,$$

etc.

Then the positive number $y - x$ would have to be less than .1, and less than .01, and less than .001, etc.;

that is, it would be less than $1/10$, less than $1/100$, less than $1/1000$, etc.

It would be infinitesimal.

But Archimedes tells us that no positive real numbers are infinitesimal.

So, we now have a way to understand $.3333\ldots$ that doesn't involve understanding infinite sums.

$.3333\ldots$ is the unique x satisfying

$$.3 < x < .4,$$

$$.33 < x < .34,$$

$$.333 < x < .334,$$

etc.

Now it's time to apply my approach to $0.999\ldots$ and see if it works.

I'd like to say that $0.999\ldots$ is the unique x satisfying

$$0.9 < x < 1.0,$$

$$0.99 < x < 1.00,$$

$$0.999 < x < 1.000,$$

etc.

Is there a problem with this?

There's no such number!

If there were such an x, $1 - x$ would have to be positive (since $x < 1$) but it'd be less than .1, and less than .01, and less than .001, etc., so it'd be infinitesimal, and there are no infinitesimal positive real numbers.

So here's a better definition of $.999\ldots$: it's the unique real number x satisfying

$$0.9 \leq x \leq 1.0,$$

$$0.99 \leq x \leq 1.00,$$

$$0.999 \leq x \leq 1.000,$$

etc.

Equivalently, $.999\ldots$ is the unique real number x satisfying

$$9/10 \leq x \leq 10/10$$

$$99/100 \leq x \leq 100/100,$$

$$999/1000 \leq x \leq 1000/1000,$$

etc.

In "grownup" mathematics, the definition of $.abc\ldots$ is the limit of the infinite series $x = a/10 + b/100 + c/1000 + \ldots$.

If and when you study advanced calculus, you may see a proof that this limit always exists, that it satisfies the conditions

$$a/10 \leq x \leq (a+1)/10,$$

$$(10a+b)/100 \leq x \leq (10a+b+1)/100,$$

$$(100a+10b+c)/1000 \leq x \leq (100a+10b+c+1)/1000,$$

etc., and that it's the only real number satisfying those conditions.

So the middle-school definition of $.999\ldots$ that I've shown you is equivalent to the calculus definition, but you need advanced calculus to show that they're equivalent.

This isn't the end of the story, because the real number system isn't the only number system mathematicians use.

There are many others, and some of them have infinitesimals, such as the nonstandard real numbers and the surreal numbers.

In both the nonstandard real numbers and the surreal numbers, there are INFINITELY many numbers satisfying the conditions

$$.9 \leq x \leq 1.0,$$

$$.99 \leq x \leq 1.00,$$

$$.999 \leq x \leq 1.000,$$

etc.

I want to end by talking about a crazy number system called the 10-adic numbers in which you have numbers that are infinite to the LEFT rather than the right.

(So for instance "backwards pi" $= \ldots 5141.3$ is a 10-adic number. But to keep things simple, let's not have any digits to the right of the decimal point, and while we're at it let's not have a decimal point at all.)

In this system, the number $\ldots 999$ is equal to -1.

Can anyone see why that might make sense?

Try adding 1 to it.

We keep writing 0 and carrying a 1.

In the end, all we have is 0's.

What is the number $\ldots 1111$ equal to in this system?

$\ldots 1111 = -1/9$, just as we saw half an hour ago!

The seemingly bogus proof that I showed you then is actually a valid proof that if we're in a number system in which $\ldots 1111$ is meaningful, then it had better equal $-1/9$.

I want to introduce you to my favorite 10-adic number.

You can meet it for yourself on your calculator.

Square 5. Square the result. Keep squaring.

Your numbers "converge 10-adically" to the 10-adic number $x = \ldots 12890625$ satisfying the equation $x^2 = x$.

Of course, 0 and 1 satisfy this equation too.

Can you find another 10-adic number y that satisfies $y^2 = y$?

$y = \ldots 87109376$ has the property that $y^2 = y$.

Here's a quicker way to get it:

Notice that if x is its own square, that is, if $x^2 = x$, then $(1-x)^2 = 1 - x - x + x^2 = 1 - x$, so $1 - x$ is its own square too!

And in fact y is $1 - x$.

Check: Add 12890625 and 87109376.

If you like this approach to math, you might enjoy my blog, Mathematical Enchantments, though it's intended more for high school students and beyond.

http://mathenchant.wordpress.com
http://mathenchant.org

You can also follow me on Twitter:

https://twitter.com/JimPropp

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Thanks for listening!