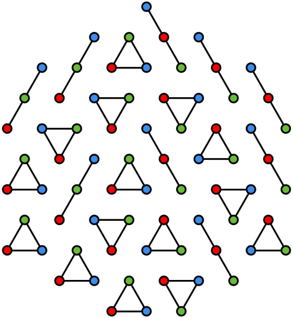
Tiling problems, old and new

James Propp, UMass Lowell
Rutgers Math Colloquium
March 30, 2022



Slides for this talk are at http://jamespropp.org/rutgers22.pdf

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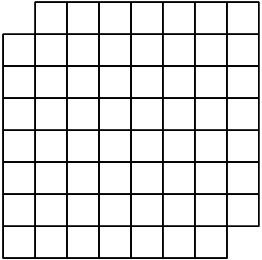
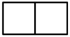
A classic puzzle

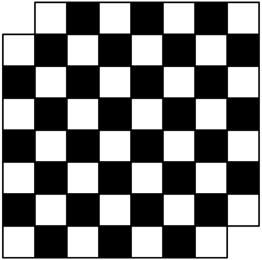


$\stackrel{?}{=}$ 31



Show that if two opposite 1-by-1 corner-squares of an 8-by-8 square are removed, the remaining region cannot be tiled by 1-by-2 and 2-by-1 rectangles.

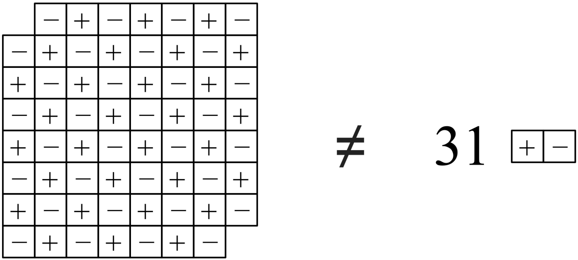
The classic proof



$\neq \quad 31$

Each tile has one white square and one black square.
But the region being tiled has unequal numbers of black and white squares.

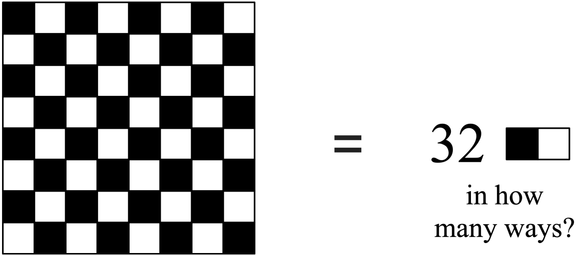
A variant proof



Each tile has total weight zero.

But the region being tiled has nonzero total weight.

A less famous problem



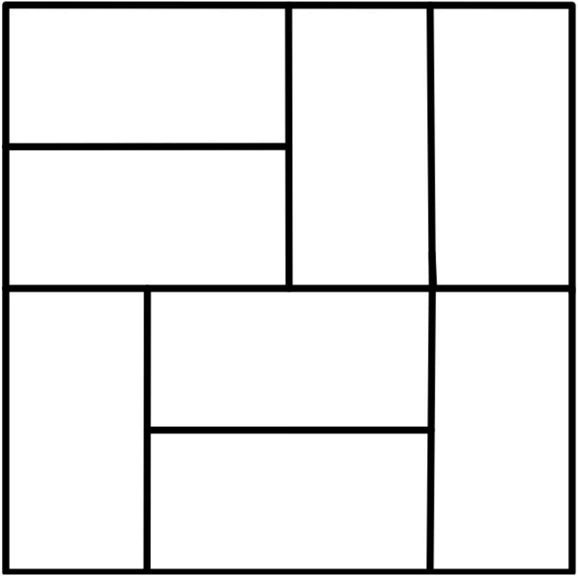
How many tilings are there if you **don't** remove those two opposite corner-squares?

Physics meets graph theory

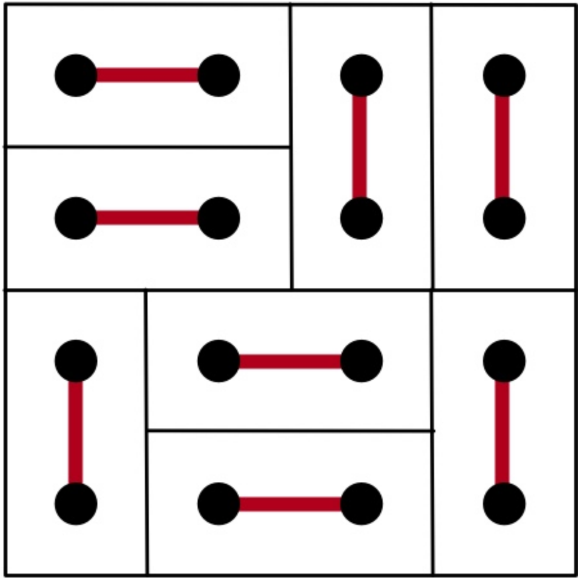
This is an example of a dimer problem, first considered by physicists, at the intersection of graph theory and enumerative combinatorics.

The physicists Temperley and Fisher, and the physicist independently Kasteleyn, solved the problems simultaneously and independently in 1961.

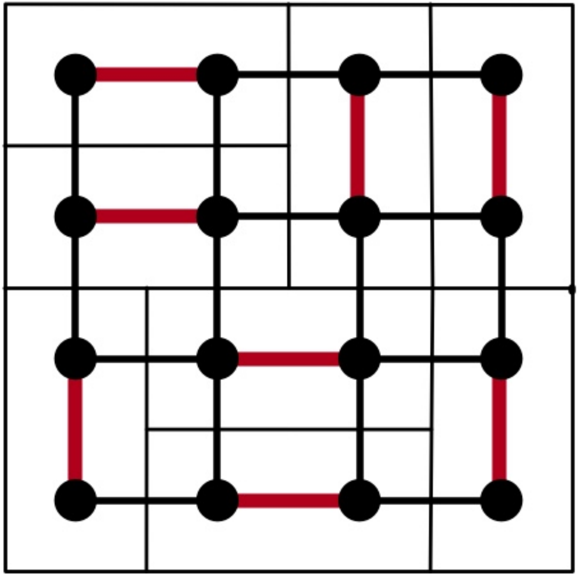
From dominos



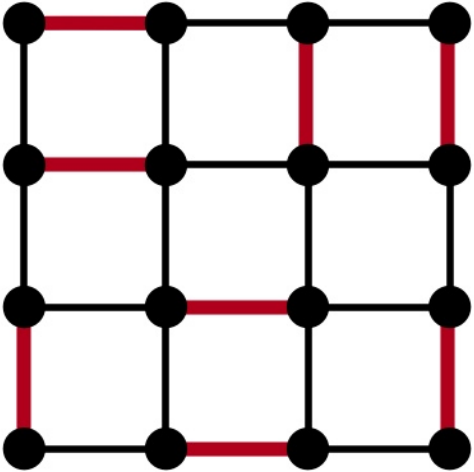
From dominos to dimers



From dominos to dimers on a



From dominos to dimers on a square grid-graph



Dimers

A **dimer** is an edge joining two vertices in a graph.

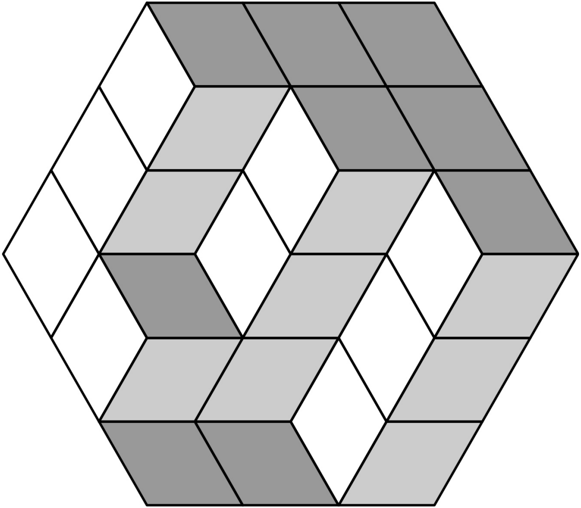
We say that a collection of dimers in a graph is a **dimer cover** if every vertex belongs to exactly one dimer in the collection.

Graph theorists call dimer covers **perfect matchings**.

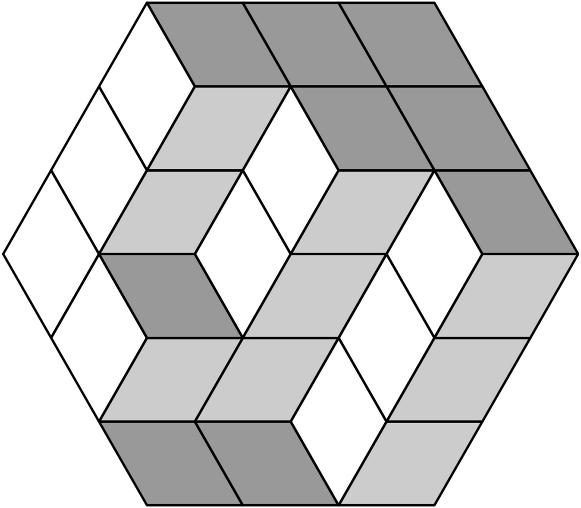
We say that the tiling model and the dimer model are duals.

Lozenges

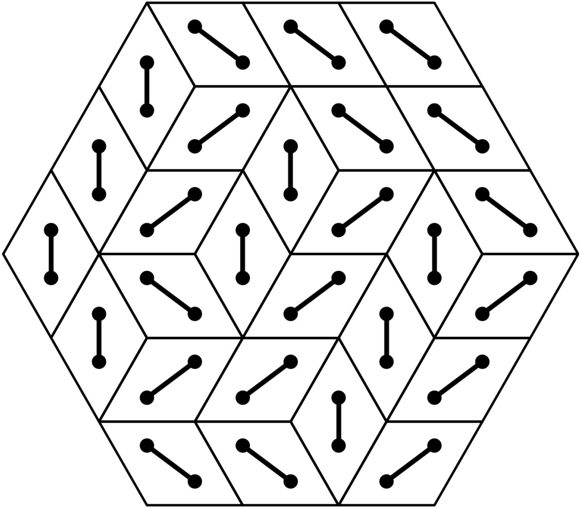
Just as dominos are made of two squares joined edge to edge, a **lozenge** (or **calisson**) is made of two equilateral triangles joined edge to edge.



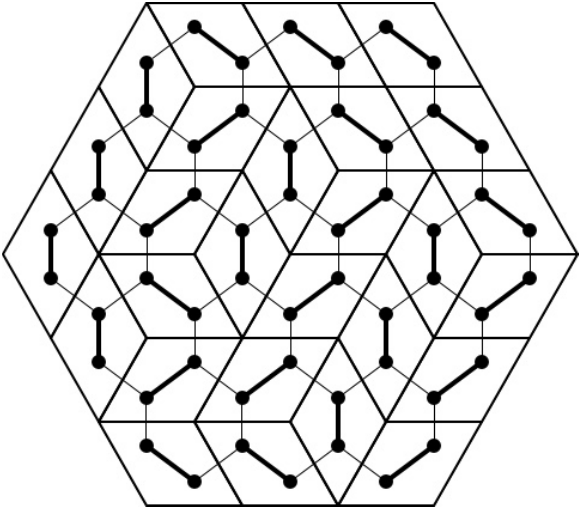
From lozenges



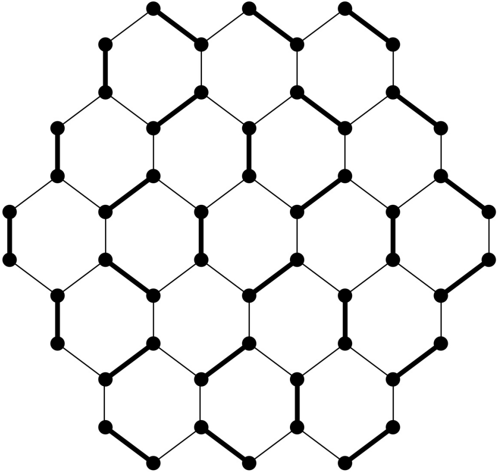
From lozenges to dimers



From lozenges to dimers on a



From lozenges to dimers on a hexagonal grid-graph



Counting dimer covers

Temperley-Fisher and Kasteleyn (1961): The number of domino tilings of a $2n$-by-$2n$ square is

$$\prod_{j=1}^{n}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{2n+1}+4\cos^2\frac{k\pi}{2n+1}\right)$$

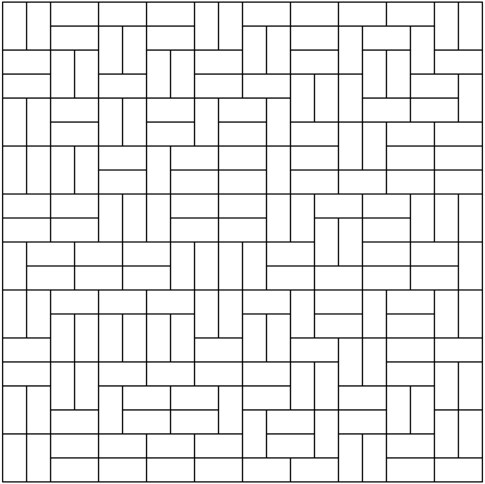
MacMahon (1896, sort of), and MacDonald (1967?): The number of lozenge tilings of a regular hexagon of side-length n is

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\prod_{k=1}^{n}\frac{i+j+k-1}{i+j+k-2}$$

Both formulas exhibit quadratic-exponential growth.

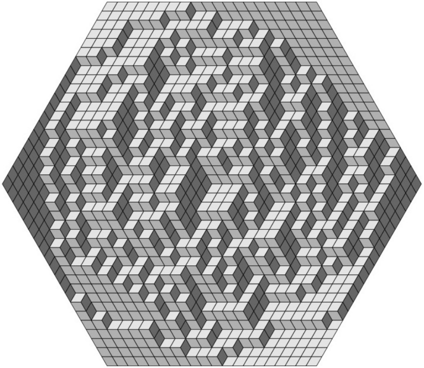
Randomness

A random domino tiling of a large square (chosen from the uniform distribution) looks more or less the way you'd expect it to.



Long-range order

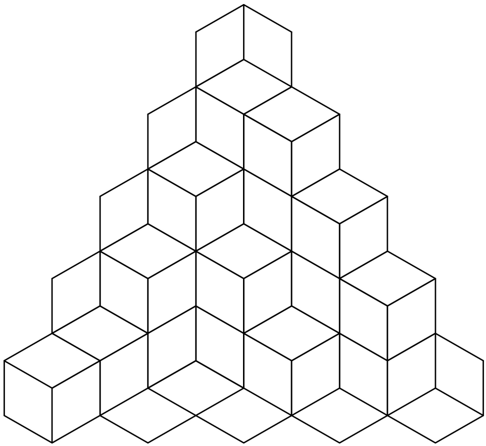
But a random lozenge tiling of a large hexagon doesn't look very random near the corners!



Why the different behavior?

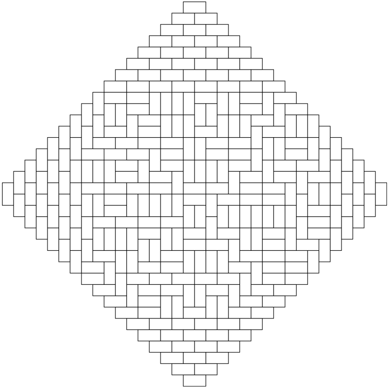
It's not about the tiles; it's about the shape of the boundary.

Here's a different boundary for lozenge-tilings that makes all the long-range order disappear.



The Aztec diamond

And here's a boundary for domino-tilings that gives rise to long-range order.



Elkies-Kuperberg-Larsen-Propp (1992): The Aztec diamond of order n has exactly $2^{n(n+1)/2}$ domino tilings.

Frozen and temperate regions

For both dominos in an Aztec diamond and lozenges in a hexagon, there are **frozen regions** near the corners within which nearly all possible tilings agree.

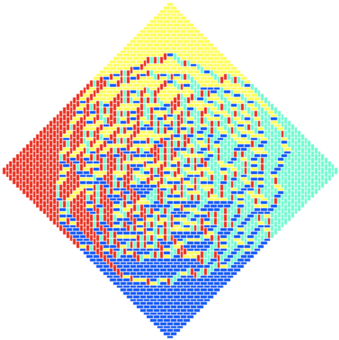
That is, for a possible tile-placement near the corner, either $0.000\ldots$ or $99.999\ldots$ percent of the tilings of size n will include that tile, when n is large.

Putting it differently, if you put a tile-shaped hole in the region, the number of tilings of the new region is either much smaller than the number of tilings was before or it is essentially equal to the number of tilings before.

Meanwhile, in the **temperate region** in the middle, local randomness reigns.

Arctic circle theorem for Aztec diamonds

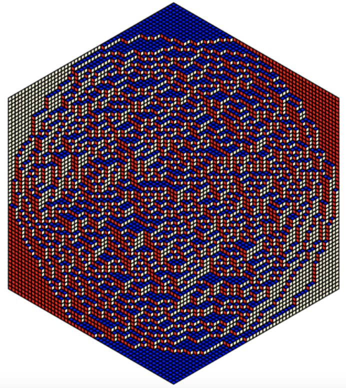
Cohn-Elkies-Propp (1996): The boundary of the frozen region for random domino tilings of an Aztec diamond of order n is asymptotically a circle.



(Mathologer made a fun video about this.)

Arctic circle theorem for hexagons

Cohn-Larsen-Propp (1998): The boundary of the frozen region for random lozenge tilings of a hexagon of order n is asymptotically a circle.



(image by Morales, Pak, and Panova)

Conway, Lagarias, and Thurston

Part of what launched this work on tilings was an article by Conway and Lagarias along with a follow-up article by Thurston, posing and solving tiling existence problems that can't be solved using coloring arguments and weighting arguments.

Tiling with Polyominoes and Combinatorial Group Theory

J. H. Conway

Princeton University, Princeton, New Jersey

AND

J. C. Lagarias

AT&T Bell Laboratories, Murray Hill, New Jersey

Communicated by Andrew Odlyzko

Received May 3, 1988

When can a given finite region consisting of cells in a regular lattice (triangular, square, or hexagonal) in $\mathbb{R}^2$ be perfectly tiled by tiles drawn from a finite set of tile shapes? This paper gives necessary conditions for the existence of such tilings using *boundary invariants*, which are combinatorial group-theoretic invariants associated

Conway's Tiling Groups

William P. Thurston, *Princeton University*

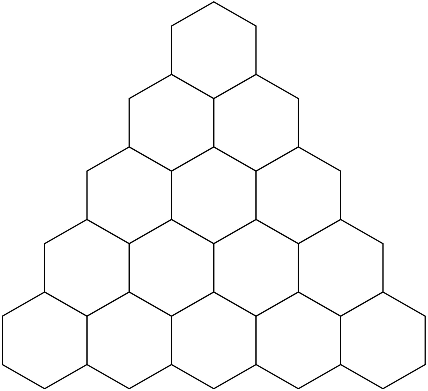
Dr. Thurston's numerous distinctions and honors include the Oswald Veblen Prize in Geometry, the Alan T. Waterman Award of the NSF, and the Fields Medal. His Ph.D. is from Berkeley, in 1972.

1. Introduction

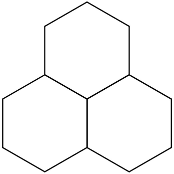
John Conway discovered a technique using infinite, finitely presented groups that in a number of interesting cases resolves the question of whether a region in the plane can be tessellated by given tiles. The idea is that the tiles can be interpreted as describing relators in a group, in such a way that the plane region can be tiled, only if the group element which describes the boundary of the region is the trivial element 1.

A trihex tiling problem

"Can you tile a T_n with T_2's?"

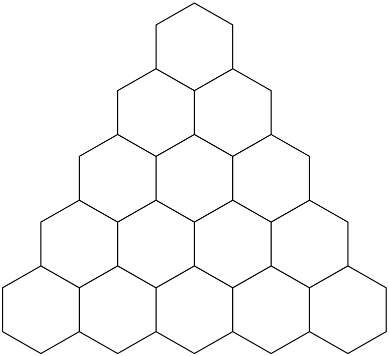


$\stackrel{?}{=}$ 5

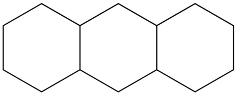


Another trihex tiling problem

"Can you tile a T_n with L_3's?"



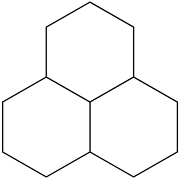
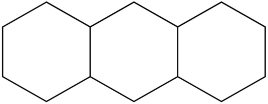
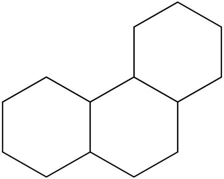
$\stackrel{?}{=}$ 5



Renaming the tiles

Stones, bones, and phones:



Conway and Lagarias' results on trihex tilings

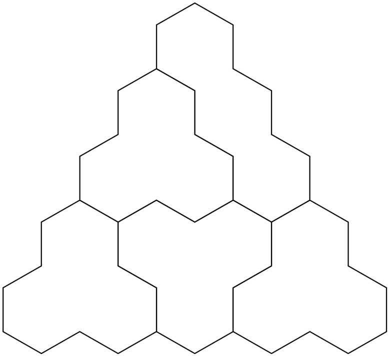
Conway and Lagarias (1990): The "honeycomb triangle" with $n \geq 1$ hexagons on a side can be tiled by stones if and only if n is congruent to 0, 2, 9, or 11 (mod 12).

Conway and Lagarias (1990): The honeycomb triangle with $n \geq 1$ hexagons on a side **cannot** be tiled by bones.

Their proof introduced a brilliant and original new method to the study of tilings: **boundary invariants** coming from combinatorial group theory (and a bit of combinatorial topology, in Thurston's treatment).

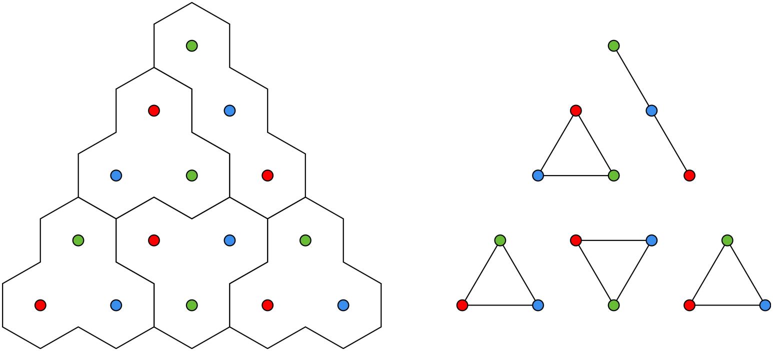
Allowing both tiles

Incidentally, it's not hard to tile a honeycomb triangle with n hexagons on a side if we allow both stones and bones, as long as n is congruent to 0 or 2 (mod 3) so that the number of hexagons is a multiple of 3.



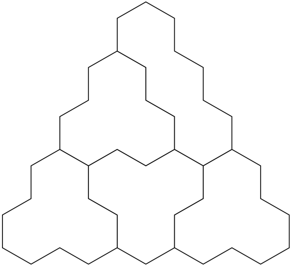
Trihexes and trimers

Physicists might prefer to think of trihex tilings as trimer covers of a finite subgraph of a triangular lattice.



The Conway-Lagarias invariant

For a simply-connected honeycomb region R and a tiling T of R by stones and bones, let $I(T)$ be the number of upward-pointing stones minus the number of downward-pointing stones. E.g., $I(T) = 3 - 1 = 2$ for the tiling T shown below.



Conway and Lagarias showed that $I(T)$ depends only on the region R, not the tiling T!

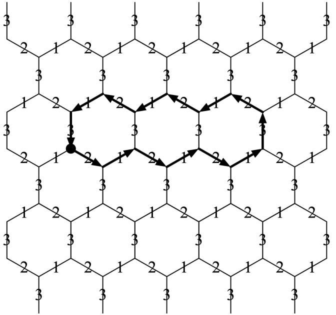
How they did it

One way of describing what Conway and Lagarias did (closer in some ways to Thurston's treatment) is to assign areas to tiles in a weird way, by transporting the whole tiling to a parallel universe and measuring areas there.

Here area means the **algebraic area** enclosed by a plane curve: it's positive if the curve encloses a region counterclockwise, it's negative if the curve encloses a region clockwise, and it could be zero if the curve crosses itself (e.g., a symmetrical figure-eight curve encloses signed area zero).

From tiles to words

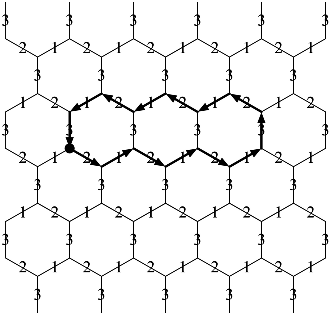
Label the edges in the honeycomb 1, 2, and 3 as shown. Then we read off the labels as we travel around a tile counterclockwise. We never see two identical labels in a row.



2 1 2 1 2 1 3 2 1 2 1 2 1 3

From words to weird words

Create a new word that **winds** where the original word **weaves** and vice versa. (Weaving means *iji*; winding means *ijk* with $k \neq i$. Alternative terminology: tacking and turning.)

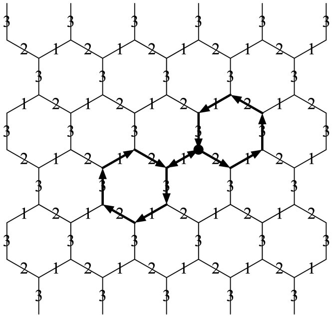


2 1 2 1 2 1 3 2 1 2 1 2 1 3

2 1 3 2 1 3 1 3 1 2 3 1 2 1

From weird words to weird paths

Now take that weird word and turn it into a weird path, reversing the procedure we used to turn the original path into the original word.



2 1 2 1 2 1 3 2 1 2 1 2 1 3

2 1 3 2 1 3 1 3 1 2 3 1 2 1

Punchline

Bones turn into figure-eights that enclose signed area

$$0;$$

Δ-stones turn into closed paths enclosing signed area

$$+3;$$

and ∇-stones turn into closed paths enclosing signed area

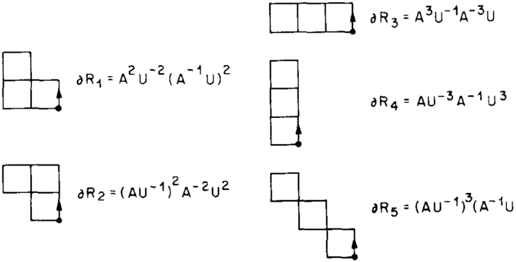
$$-3.$$

So the number of upward-pointing stones minus the number of downward-pointing stones equals the signed area enclosed by the weirdification of the boundary of the original region being tiled, which clearly doesn't depend on the tiling.

"Yes, but how many tilings are there?"

In how many ways can one tile a honeycomb triangle with n hexagons on a side by stones and bones?

David DesJardins wrote a program to count tilings of regions like this, although first we need to apply a skewing, as in Conway and Lagarias' article:



The Online Encyclopedia of Integer Sequences

After running his program for small n, David entered his numbers into the OEIS and discovered others had already explored the question:

The OEIS is supported by the many generous donors to the OEIS Foundation.

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A334875 Number of distinct tribone tilings of an n-triangle. 0

1, 0, 1, 0, 0, 0, 0, 0, 0, 2, 0, 8, 12, 0, 72, 0, 0, 0, 0, 0, 0, 185328, 0, 4736520, 21617456, 0, 912370744, 0, 0, 0, 0, 0, 0, 3688972842502560, 0, 717591590174000896, 9771553571471569856, 0, 3177501183165726091520, 0, 0, 0, 0, 0 (list; graph; refs; listen; history; edit; text; internal format)

OFFSET 0,10

COMMENTS This sequence was requested to be added by the author of the link Code Golf challenge. It is based on the work of J. H. Conway, who proved that n = 12k + 0,2,9,11 if and only if T(n) can be tiled (i.e., exactly covered without overlapping) by tribones.

Changing the question

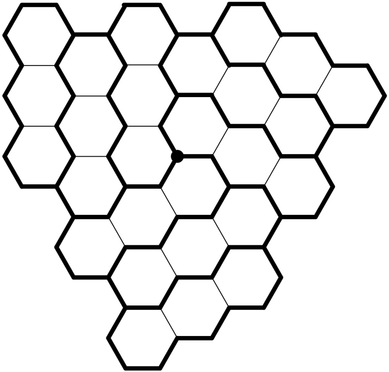
Nobody's found a formula for the terms of https://oeis.org/A334875 counting stones-and-bones tilings of honeycomb triangles, but a different class of regions has led to some interesting conjectures.

For motivation, watch the video of my talk at the Combinatorics and Arithmetic for Physics workshop hosted last year at IHES.

Essentially, benzels are to honeycomb triangles as Aztec diamonds are to squares; they're designed to be just-barely-tileable at the boundary so that large-scale structures are likely to form and propagate into the interior of the region.

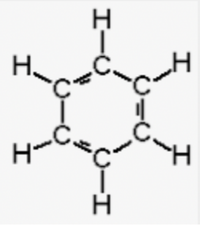
What do benzels look like?

Benzels form a two-parameter family, with parameters a and b satisfying $a \leq 2b - 2$ and $b \leq 2a - 2$. Here is the 5,7-benzel, tiled by bones:

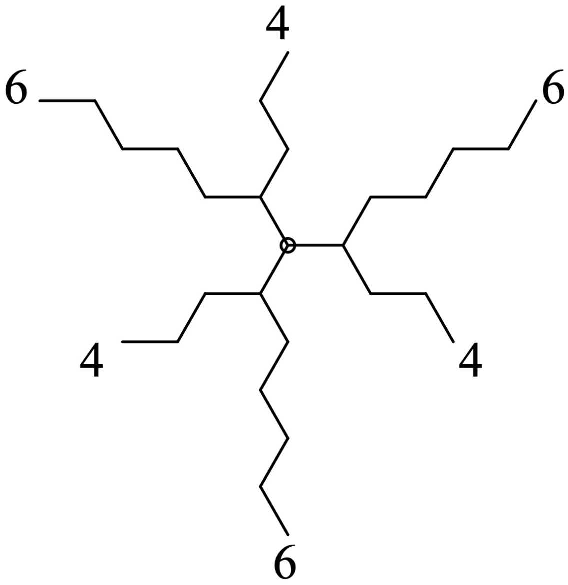


(Note that I've rotated my hexagonal cells relative to the kinds used earlier in my talk.)

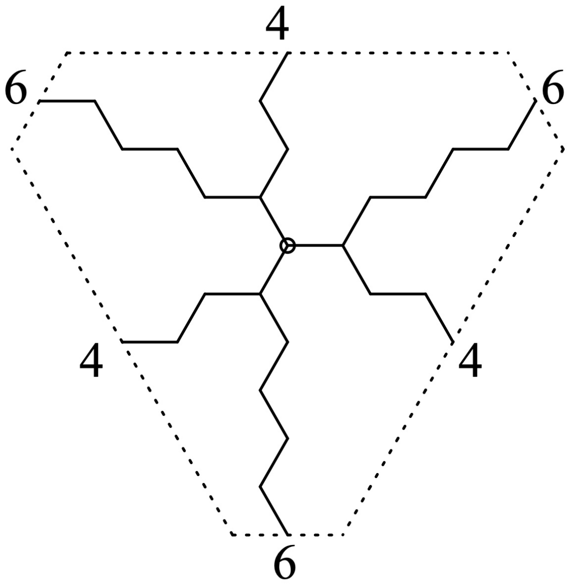
Why "benzel"?



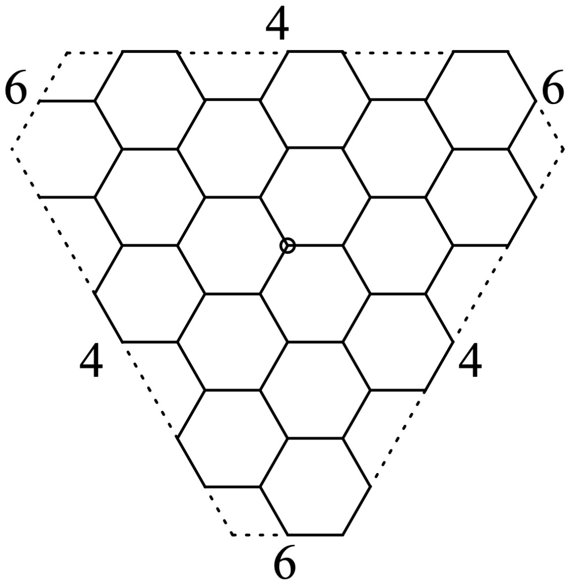
Building a benzel, step 1



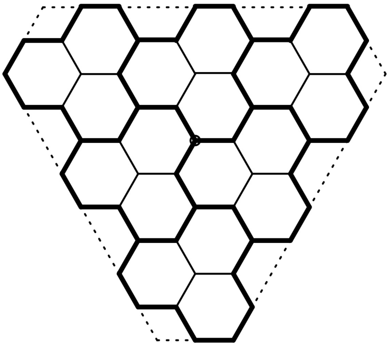
Building a benzel, step 2



Building a benzel, step 3



The 4,6-benzel, tiled by stones



All the stones point the same way, so the Conway-Lagarias invariant tells us it has only one tiling.

I believe (but have not yet proved) that the a, b benzel can be tiled by stones alone exactly when $a + b$ is 1 (mod 3).

Tiling with bones alone: audience quiz

On the other hand, hardly any benzels can be tiled by bones alone. For instance, with $a, b \leq 10$, only the 5,7-benzel (and the 7,5-benzel) can be tiled by bones alone.

But there is another (with larger a, b). Any guesses?

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So can the 12,15-benzel.

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So can the 12,15-benzel.

But there is another. Any guesses?

So can the 22,26-benzel.

What's the pattern?

Paired pentagonal numbers

Theorem (March 2022): The a, b-benzel can be tiled by bones alone only if a, b are paired pentagonal numbers $n(3n \pm 1)/2$.

Proof sketch:

1. Compute the Conway-Lagarias invariant $I(R)$ (three cases)
2. Show that the only feasible case is $a + b \equiv 0 \pmod 3$, with $I(R) = (a + b - 3a^2 + 6ab - 3b^2)/6$.
3. Set $I(R) = 0$. Solving for b in terms of a, show that $24b + 1$ must be a perfect square.
4. Show that this happens only when a and b are paired pentagonal numbers.

1 kind of stone, 2 kinds of bones

Allow 1 (of the 2) kinds of stones and 2 (of the 3) kinds of bones. How many tilings are there? With $3 \leq a, b \leq 15$:

2	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	0	0	0	0	0	0	0	0	0	0
0	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	8	0	0	0	0	0	0	0	0	0
0	0	0	0	0	8	0	0	0	0	0	0	0
0	0	0	0	8	0	0	0	0	0	0	0	0
0	0	0	0	0	0	48	0	0	0	0	0	0
0	0	0	0	0	0	0	0	48	0	0	0	0
0	0	0	0	0	0	0	48	0	0	0	0	0
0	0	0	0	0	0	0	0	0	384	0	0	0
0	0	0	0	0	0	0	0	0	0	0	384	0
0	0	0	0	0	0	0	0	0	0	384	0	0
0	0	0	0	0	0	0	0	0	0	0	0	3840

Double factorials

Conjecture: When a and b both equal $3n$, or when one is $3n+1$ and the other is $3n+2$, the number of tilings of the a, b-benzel with 2 kinds of bones and 1 kind of stone allowed is $2 \times 4 \times \cdots \times 2n$.

2 kinds of stones, 2 kinds of bones

Allow 2 (of the 2) kinds of stones and 2 (of the 3) kinds of bones. How many tilings are there? With $3 \le a, b \le 10$:

2	1	1	1	1	1	1	1
1	4	6	1	1	1	1	1
1	6	1	16	22	1	1	1
1	1	16	48	1	68	90	1
1	1	22	1	224	512	1	304
1	1	1	68	512	1	3360	6736
1	1	1	90	1	3360	15360	1
1	1	1	1	304	6736	1	168960

Hiding along one diagonal are 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, ..., which I didn't recognize but the OEIS did: these are the large Schroder numbers.

A job for superseeker

Another diagonal gives a sequence 2, 48, 15360, 65601536, 3737426853888, ... whose nth entry is an enormous number with no prime factor larger than $4n$.

It should be given by a product formula like the MacMahon/MacDonald formula.

OEIS didn't recognize these terms.

"This looks like a job for Superseeker!"

A job for the DOMINO listserv

Superseeker wasn't able to crack it, so I posted to the DOMINO listserv, and David DesJardins was able to find the pattern:

```
I did an exhaustive search for rational functions of k with simple
factorizations to try to match A(k+2)*A(k)/A(k+1)^2 and I got a hit:

A(k) = {2, 48, 15360, 65601536, 3737426853888, 2839095978202497024,
28748176693620694822420480, 387952004963238149100725600256000}

A(k+2)*A(k)/A(k+1)^2
= {40/3,1001/75,146880/11011,43681/3276,12516140/938961,704700/52877}
= 256 (2 k+3)^2 (4 k+1) (4 k+3)^2 (4 k+5) /
  (27 (3 k+1) (3 k+2)^2 (3 k+4)^2 (3 k+5)) for k={1,2,3,4,5,6}
```

It worked for the next value of k as well, so it seems likely to be right. But we have no proof.

2 kinds of stones, 3 kinds of bones

There are no conjectural product formulas, but two diagonals of the table have regular 2-adic behavior.

With $a = n$ and $b = 2n - 3$, the number of tilings goes
0,0,0,0,... (mod 2);
2,2,2,2,... (mod 4);
2,6,2,6,... (mod 8).

I conjecture that the residues mod 2^k are periodic mod 2^j for some j ("2-adic continuity").

Likewise for $a = n$ and $b = 2n - 4$.

Where does 2-adic continuity come from?

I know of just one (proved) example of 2-adic continuity arising from enumeration of tilings.

Let $T(n)$ be the number of domino tilings of a $2n$-by-$2n$ square. It was long known that $T(n)$ can be expressed as $2^n S(n)^2$ for some integer $S(n)$.

Henry Cohn showed that the sequence $S(n)$ (https://oeis.org/A065072) is 2-adically continuous.

His proof used the exact formula of Temperley-Fisher and Kasteleyn.

But in our examples we have no exact formula!

Honeycomb triangles revisited

Let's take another look at the sequence https://oeis.org/A334875 counting tilings of honeycomb triangles by stones.

The multiplicity of the prime 2 in the factorizations of the nonzero terms in this sequence are 0, 0, 1, 3, 2, 3, 4, 3, 4, 3, 5, 8, 6, 8 which seem to show an upward drift.

Conjecture: The number of tilings goes to zero 2-adically.

Mutations

Open problem: Can every stones-and-bones tiling T of a simply-connected honeycomb region be obtained from every other tiling T' by means of a sequence of "2-flips", each of which replaces 2 tiles by 2 other tiles?

If true, this would yield the invariance of the Conway-Lagarias invariant as a corollary, since it can be checked that 2-flips don't change the value of $I(T)$.

Random tilings

Preliminary results indicate that along the boundary of a benzel, a random stones-and-bones tiling is “close to nonrandom”.

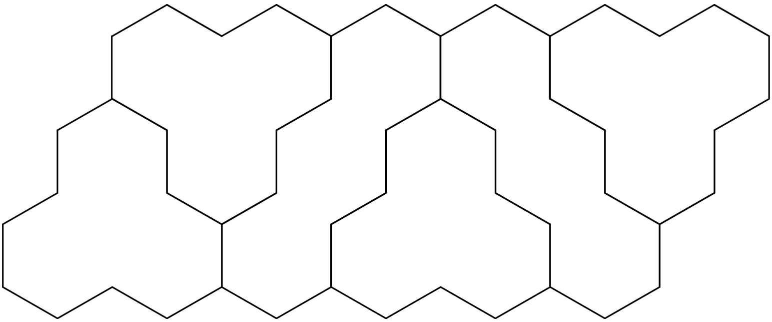
I would expect that the tilings exhibit long-range order.

But: how do you generate tilings uniformly at random? I don’t know!

An accessible counting problem

Form a honeycomb parallelogram with 3 hexagons on its short sides and n hexagons on its long sides.

Forbid bones that are parallel to the long sides of the parallelogram, but allow the other four types of tiles.



How many tilings are there?

Thanks!

That's all I got; thank you for listening!

Slides for this talk are at http://jamespropp.org/rutgers22.pdf