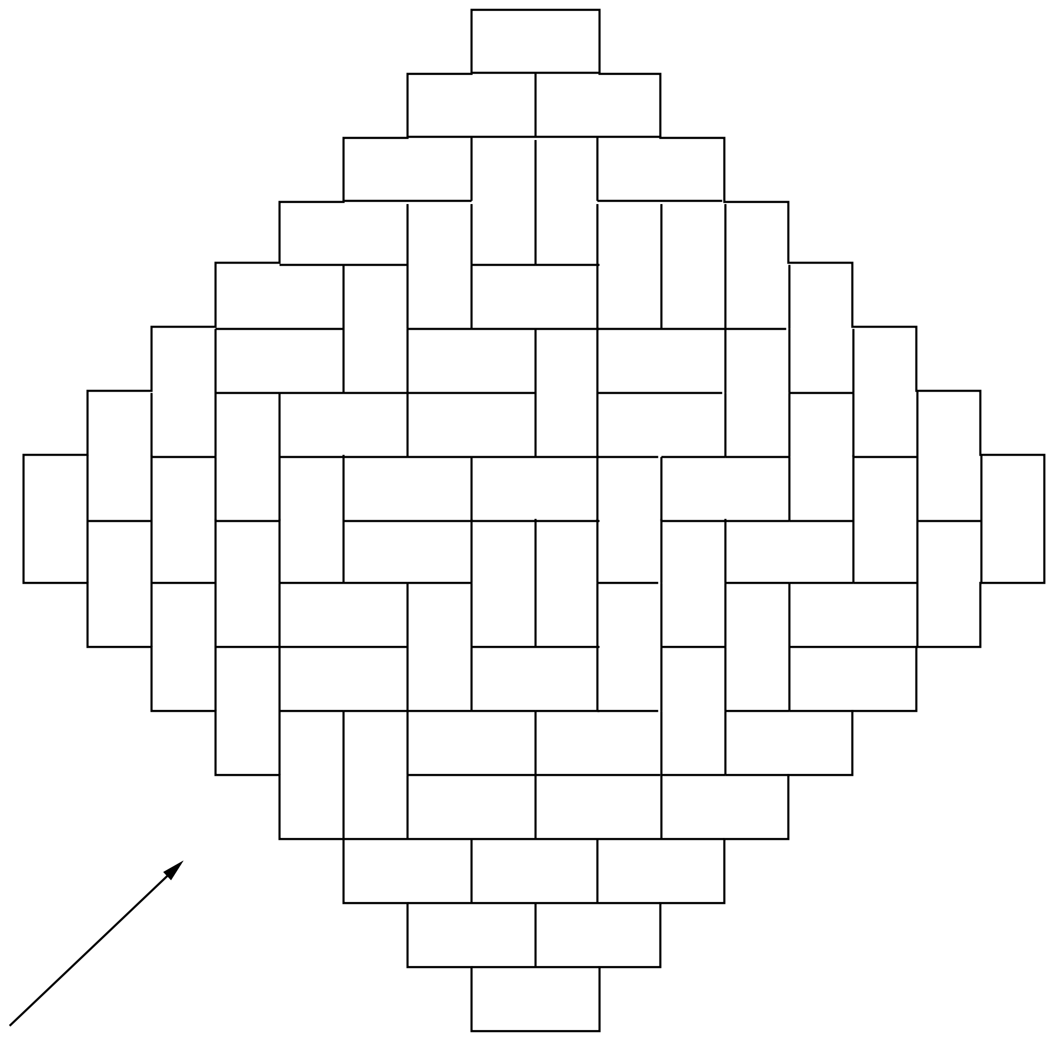
Theorem (Elkies, Kuperberg, Larsen, and Propp):
The Aztec diamond of order n **has exactly**

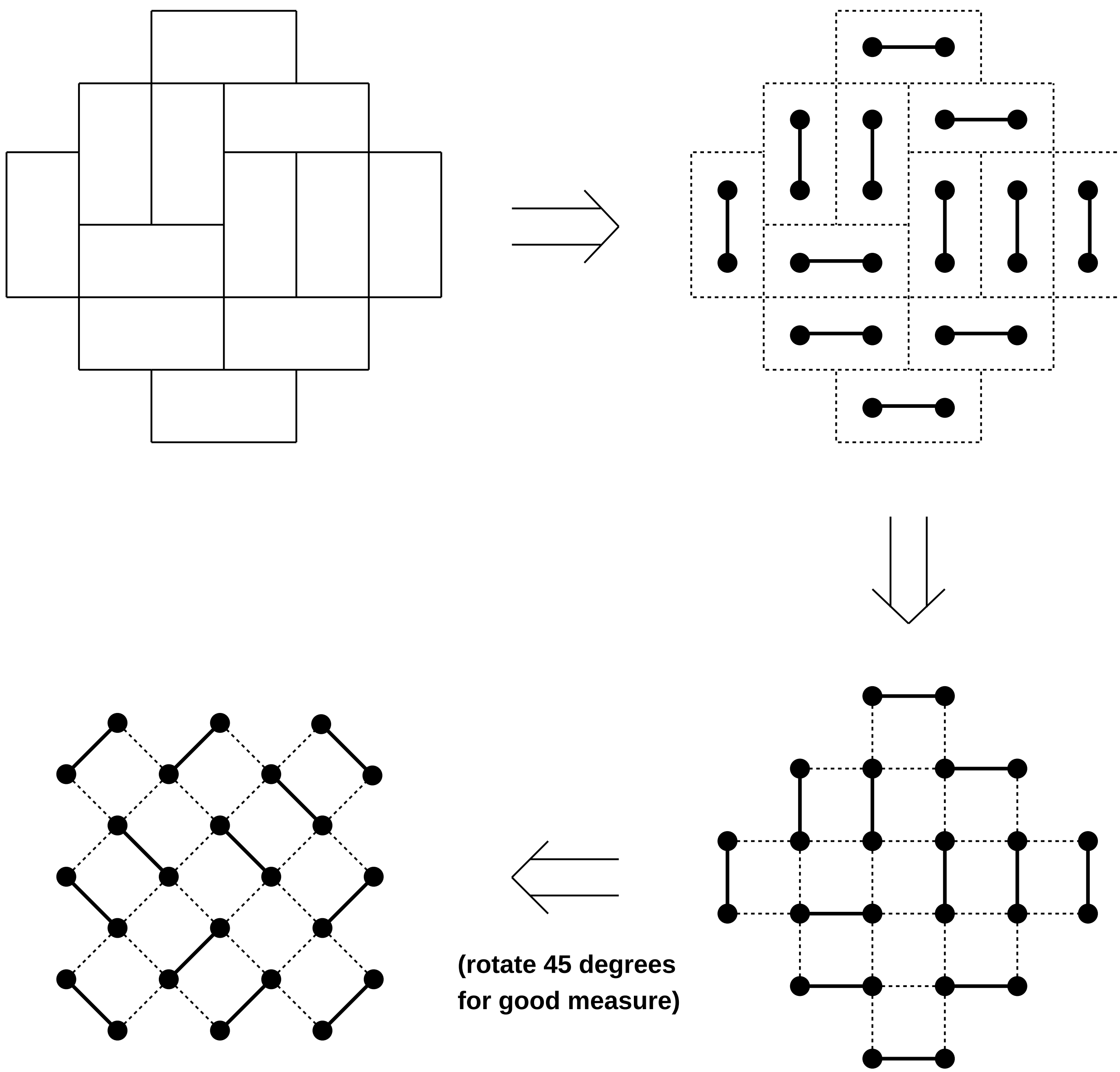
$$2^{n(n+1)/2}$$

tilings by dominoes.

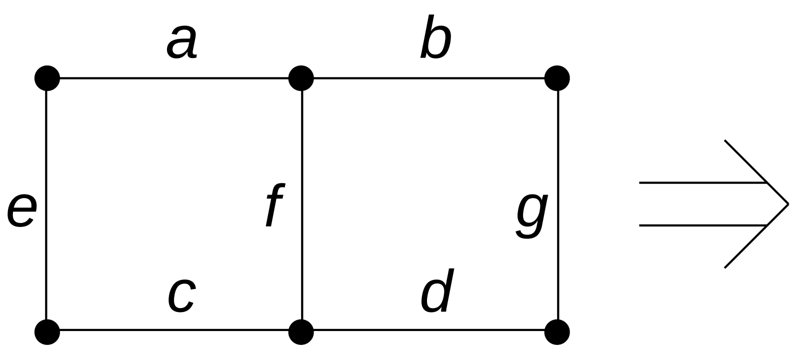


(a domino tiling of the Aztec diamond of order 8)

Duality between domino tilings and perfect matchings



Given a graph *G* with non−negative real numbers (or formal variables) called "weights" associated with its edges, we consider the sum of the weights of all the perfect matchings of *G*. For instance:

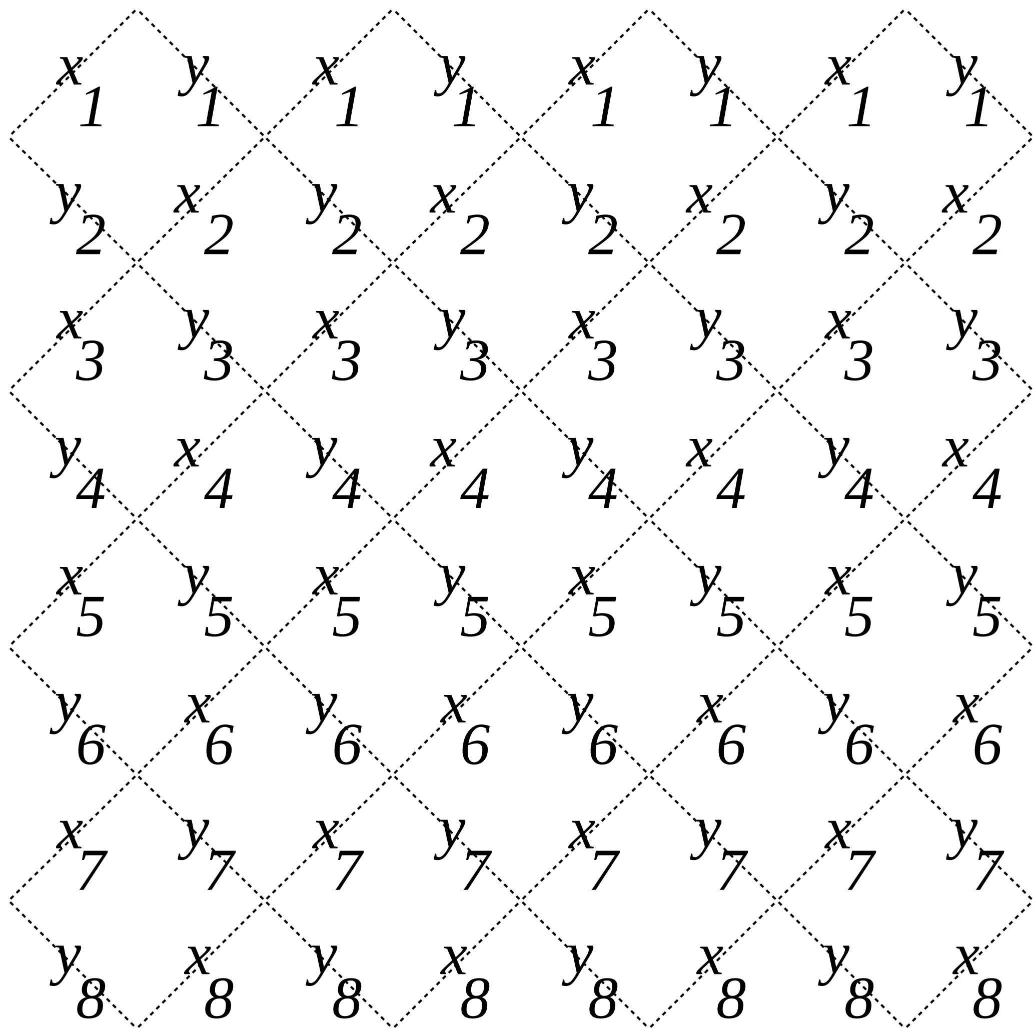


acg+bde+efg

(A *perfect matching* of a graph *G* is a collection of edges of *G* that jointly contain each vertex of *G* exactly once, and the *weight* of a collection of edges is the product of the weights of the edges.)

The number of domino tilings of a region like the Aztec diamond is the sum of the weights of all the perfect matchings of the dual graph, where each edge is given weight 1.

Richard Stanley's weighting scheme:



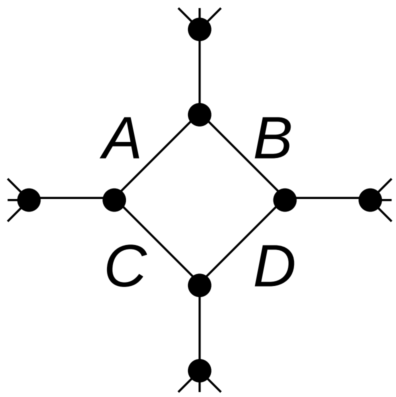
Theorem (Stanley):

Under the weighting scheme just illustrated, the sum of the weights of the matchings of the Aztec diamond of order n is

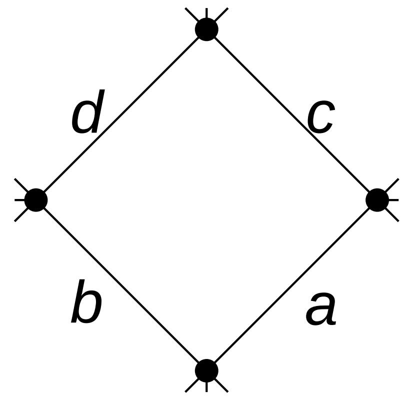
$$\prod_{\substack{1 \le i < j \le n \\ i \text{ odd} \\ j \text{ even}}} (x_i x_j + y_i y_j)$$

The local replacement lemma

The sum of the weights of the matchings of a graph that contains the patch



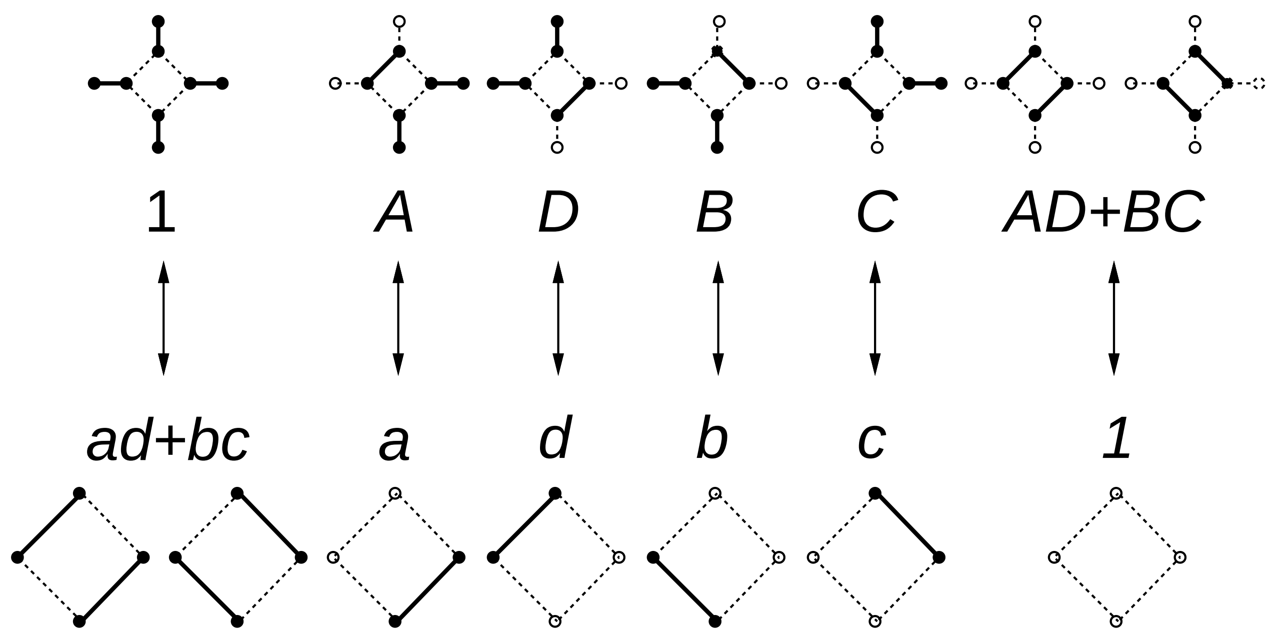
equals R times the sum of the weights of the matchings of the graph obtained by replacing the patch by



where unmarked edges have weight 1,

$R=AD+BC,\quad a=A/R,\ b=B/R,\ c=C/R,\ d=D/R.$

Check that the following two lists of numbers differ by a factor of R (note $ad+bc=1/R$):



New!

A short proof of the theorem:

Improved!

Apply local transformations* to the dual graph of the Aztec diamond of order n, obtaining the dual graph of the diamond of order $n-1$ while picking up an extra weight factor of 2^n.

*** see pages 7&8**