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SYSTEM EQUIVALENT REDUCTION EXPANSION PROCESS (SEREP)

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ABSTRACT

A new technique, the System Equivalent Reduction Expansion Process (SEREP), is presented. Originally formulated as a global mapping technique used to estimate rotational degrees of freedom for experimental modal data, SEREP has been found to provide improved accuracy in applications such as cross orthogonality checks between analytical and experimental modal vectors, linear and nonlinear forced response studies, and analytical model improvement. The theory upon which SEREP is based, as well as applications showing the advantages of this new technique are presented.

INTRODUCTION

Reduction or condensation techniques are generally employed to reduce the size of large analytical models to minimize computer time/cost or to deal with a reduced model for forced response studies. Analytical model reduction techniques may also be employed to generate a set of analytical degrees of freedom (dof) that correspond to a set of measured dof. Most reduction techniques will effect the dynamic character contained in the original full analytical model. Generally, the estimated frequencies in the reduced model are higher than those of the original model. However, a reduction technique has been developed that produces reduced models (containing arbitrarily selected dof) which preserves the dynamic character of the original full system model for selected modes of interest. This technique is called the System Equivalent Reduction/Expansion Technique (SEREP).

SEREP was originally formulated as a global mapping technique that was used to develop rotational dof for modal test data (1). Since then however, SEREP has been successfully employed in a variety of applications such as checking correlation and orthogonality between analytical and experimental modal vectors (2), linear and nonlinear forced response studies (3), and analytical model improvement (4).

SEREP provides features that other reduction techniques do not such as

- the arbitrary selection of modes that are to be preserved in the reduced system model
- the quality of the reduced model is not dependent upon the location of the selected active dof
- the frequencies and mode shapes of the reduced system are exactly equal to the frequencies and mode shapes (for the selected modes) of the full system model
- the reduction/expansion process is reversible; expanding the reduced system's mode shapes back to the full system's space, develops mode shapes that are exactly the same as the original mode shapes of the full system model

Application of SEREP is presented on a simple stucture to demonstrate the accuracy of the reduced model as well as the ability to arbitrarily select modes and arbitrarily select dof in the reduced model.

THEORY

The equations of motion for an undamped 'n' dimensional system are

$$\underset{\sim}{M}_n \, \ddot{\underline{X}}_n + \underset{\sim}{K}_n \, \underline{X}_n = \underline{0} \qquad (1)$$

where $\underset{\sim}{M}_n$ and $\underset{\sim}{K}_n$ are the (n x n) system mass and stiffness matrices, respectively; $\ddot{\underline{X}}_n$ and $\underline{X}_n$ are the (n x 1) acceleration and displacement vectors, respectively. Note: Hereinafter, matrices are denoted by a tilda (˜) underscore and vectors denoted by a bar (_) underscore.

The eigensolution of the 'n' dimensional system is based on these 'm' modal vectors is given by

$$\underline{X}_n = \underset{\sim}{U}_n \, \underline{P} \qquad (2)$$

where $\underset{\sim}{U}_n$ is the (n x m) modal matrix whose columns are made up of the 'm' modal vectors; P is the (m x 1) displacement vector in the modal coordinate system. Note, the columns of $\underset{\sim}{U}_n$ are linearly independent and therefore, the rank of $\underset{\sim}{U}_n$ is 'm'.

It is desired that a reduced model, size (a x a), be formulated from the full system model. The full system model is partitioned so as to seperate those full system dof which will be tracked in the reduced system model, from those full system dof that will not be tracked in the reduced system model. Equation (2) is rewritten

$$\underline{Xn} = \begin{Bmatrix} \underline{X}_a \\ \underline{X}_d \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{Ua} \\ \underset{\sim}{Ud} \end{bmatrix} \underline{P} \tag{3}$$

where the 'a' subscript denotes the tracked, or active dof (Adof) of the full system model; and the 'd' subcript denotes the untracked, or deleted dof of the full system model. Considering only these Adof in Equation (3) yields

$$\underline{X}_a = \underset{\sim}{U}_a \; \underline{P} \tag{4}$$

which is a description of the system response at the 'a' active dof in terms of the 'm' modal variables. The $\underset{\sim}{U}_a$ matrix, of dimension (a x m), is generally not square. Solving Equation (4) for the modal vector, $\underline{P}$, requires that a generalized inverse of $\underset{\sim}{U}_a$ be formed.

It is shown in Appendix A that for the condition 'a' is greater than or equal to 'm', the generalized inverse is of rank 'm'. In most practical applications, the number of Adof, 'a', is greater than or equal to the number modes, 'm', used in the formulation of the generalized inverse. When this is true, the generalized inverse can be written as

$$\underset{\sim}{U}_a^{\,g} = (\underset{\sim}{U}_a^{\,T} \; \underset{\sim}{U}_a)^{-1} \; \underset{\sim}{U}_a^{\,T} \tag{5}$$

where $\underset{\sim}{U}_a^{\,g}$ is the generalized inverse of $\underset{\sim}{U}_a$.

With the generalized inverse in hand, Equation (4) is solved for the modal displacement vector as

$$\underline{P} = \underset{\sim}{U}_a^{\,g} \; \underline{X}_a \tag{6}$$

It is also shown in Appendix A that when 'a' is less than 'm', the rank of the generalized inverse is 'a'. The generalized inverse is then given by

$$\underset{\sim}{U}_a^{\,g} = \underset{\sim}{U}_a^{\,T} \; (\underset{\sim}{U}_a \; \underset{\sim}{U}_a^{\,T})^{-1} \tag{7}$$

As stated in Appendix A, this produces an average solution for the 'm' modal displacement components of $\underline{P}$ in Equation (6) and is not of practical use. Our discussion, therefore will deal only with the case of 'a' greater than or equal to 'm'.

Substituting Equation (6) into Equation (2) gives an expression for the full system's displacement vector in terms of the reduced system's displacement vector.

$$\underline{X}_n = \underset{\sim}{U}_n \; \underset{\sim}{U}_a^{\,g} \; \underline{X}_a \tag{8}$$

The global mapping transformation matrix, relating the Adof to the Ndof is then defined as

$$\underset{\sim}{T}_u = \underset{\sim}{U}_n \; \underset{\sim}{U}_a^{\,g} \tag{9}$$

or

$$T_u = \begin{bmatrix} \underset{\sim}{U}_a & \underset{\sim}{U}_a^{\,g} \\ \underset{\sim}{U}_d & \underset{\sim}{U}_a^{\,g} \end{bmatrix} \tag{10}$$

Note that the subscript 'u' denotes that this transformation matrix is based on the analytical modal vector set $\underset{\sim}{U}_n$. Substituting Equation (9) into Equation (8) yields

$$\underline{X}_n = \begin{Bmatrix} \underline{X}_a \\ \underline{X}_d \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{U}_a \; \underset{\sim}{U}_a^{\,g} \\ \underset{\sim}{U}_d \; \underset{\sim}{U}_a^{\,g} \end{bmatrix} \underline{X}_a = \underset{\sim}{T}_u \; \underline{X}_a \tag{11}$$

The $\underset{\sim}{T}_u$ matrix is used to form the reduced mass and stiffness matrices as

$$\underset{\sim}{M}_a = \underset{\sim}{T}_u^{\,T} \; \underset{\sim}{M}_n \; \underset{\sim}{T}_u \tag{12}$$

$$\underset{\sim}{K}_a = \underset{\sim}{T}_u^{\,T} \; \underset{\sim}{K}_n \; \underset{\sim}{T}_u \tag{13}$$

where $\underset{\sim}{M}_a$ and $\underset{\sim}{K}_a$ are the equivalently reduced mass and stiffness matrices, respectively. The equations of motion for the 'a' dimensional, reduced system are then

$$\underset{\sim}{M}_a \; \ddot{\underline{X}}_a + \underset{\sim}{K}_a \; \underline{X}_a = \underline{0} \tag{14}$$

The eigenvalues for this Adof system equal the eigenvalues of the Ndof system which correspond to the modes employed in the formuation of $\underset{\sim}{T}_u$. Further

$$\underline{X}_a = \underset{\sim}{U}_a' \; \underline{P} \tag{15}$$

while

$$\underset{\sim}{U}_a = \underset{\sim}{U}_a' \tag{16}$$

where $\underset{\sim}{U}_a'$ is the (a x m) modal matrix formed from the eigensolution of the Adof system (see Appendix A). Finally, one can obtain the modal vector matrix of the Ndof system by

$$\underset{\sim}{U}_n = \underset{\sim}{T}_u \; \underset{\sim}{U}_a' \tag{17}$$

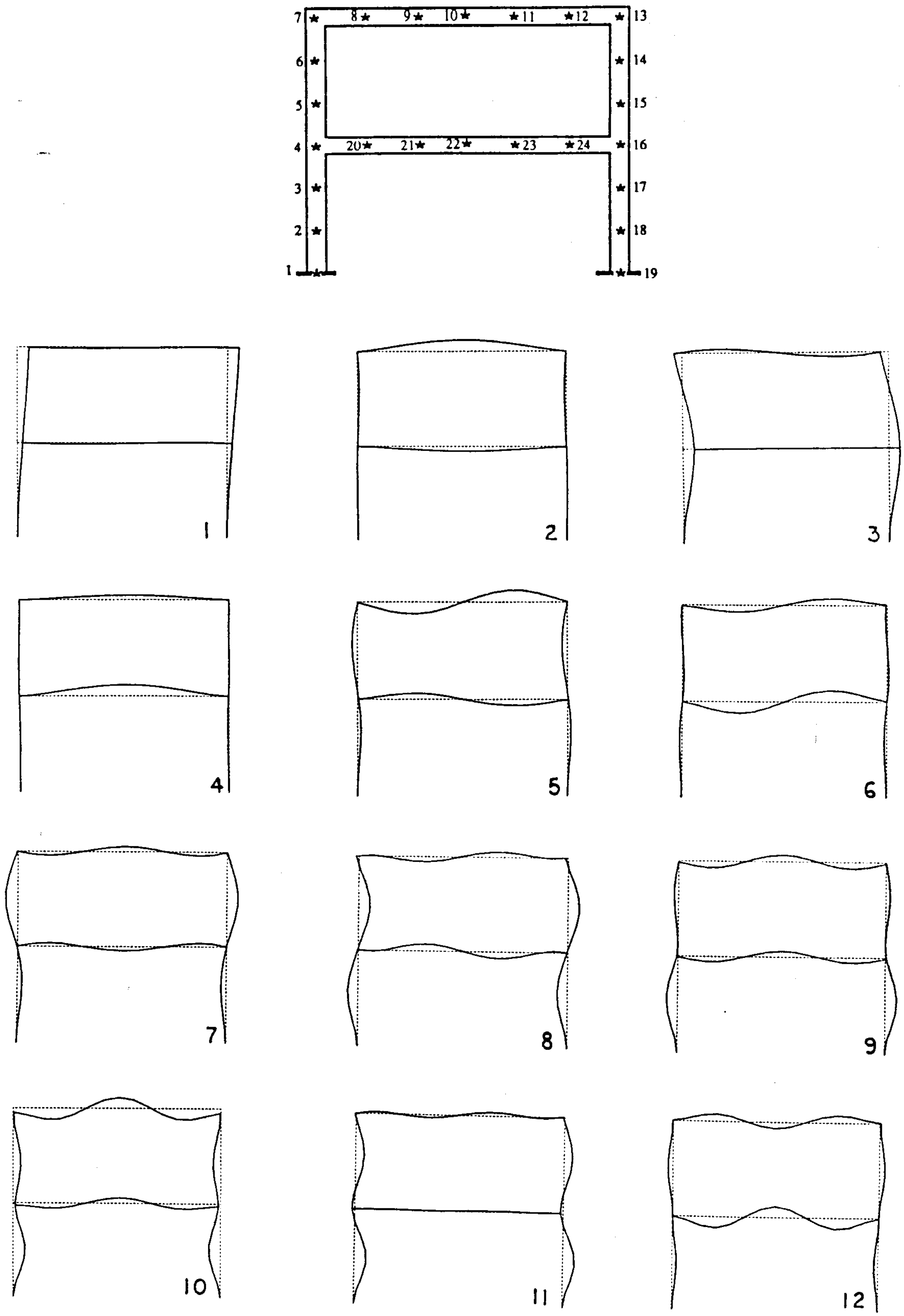
Unlike other reduction techniques, SEREP is a reversible process.

Comparison of SEREP and Guyan Reduction

Guyan reduction (6), commonly refered to as static condensation, is a technique widely used for the reduction of large analytical models. (See reference (7) for a more complete comparison of various expansion/reduction techniques.)

In Guyan reduction, the relationship between the active and deleted dof can be written as

$$\begin{matrix} \underline{X}_a \\ \underline{X}_d \end{matrix} = \begin{bmatrix} \underset{\sim}{I} \\ \underset{\sim}{t}_s \end{bmatrix} \underline{X}_a = \underset{\sim}{T}_s \; \underline{X}_a \tag{18}$$



FRAME MODEL AND MODE SHAPES

FIGURE 1

where

$$t_s = - K_{dd}^{-1} K_{da} \tag{19}$$

and T_s is referred to as the transformation matrix.

It should be noted that the transformation matrix, T_s, is based solely on the stiffness matrix. Therefore, the inertial forces of the full system are not preserved when the full system is mapped down to a reduced space while using the Guyan reduction process. Note further though that the SEREP mapping matrix, T_u, is based on the analytical modal vector set which inherently contains information concerning the inertial forces.

This Guyan projection matrix, T_s, is used to form the reduced mass and stiffness matrices K_a and M_a as well as to expand the reduced variable to the full set of Ndof using

$$X_n = T_s \; X_a \tag{20}$$

The eigensolution for this Guyan reduced system will provide a set of estimates for the first 'a' eigenvalues of the Ndof system. Also, the modal matrix formed from the eigensolution of the Adof system, $U_a{}'$, is only an approximation of the modal matrix, U_a, that is formed from the first 'a' eigenvectors of the full system. Further, the quality of the estimation of the Ndof system's eigenvalues, and modal matrix depends on the selection of the active dof that comprise the reduced system.

Note that SEREP produces a reduced model whose frequencies and modes shapes are exactly the same as the original model for the modes preserved in the process. Also, the accuracy of the SEREP reduced models is not dependent upon the dof that are chosen for the reduced system model.

APPLICATIONS

Several simple models are investigated in order to demonstrate the unique features of SEREP which are

- the exactness of the reduction technique
- the arbitrary selection of modes included in the reduced model
- the arbitrary selection of dof included in the reduced model

The six models used to study the SEREP process are described in the following paragraphs.

Reference Frame Model

A simple planar frame model model was used for all cases studied. The finite element model was comprised of 24 nodes and 24 planar beam elements with 3 dof per node. This 72 dof model is considered the reference to which all the reduced models studied in the following examples will be compared. Figure 1 shows the frame model as well as some of its typical mode shapes.

Exact System Reduction

To examine the point that the reduced model reproduces the full model in an exact sense, the basic FRAME structure was reduced to a smaller model and a comparison of actual full system's natural frequencies and reduced system's natural frequencies was made.

SEREP is used to reduce the full system model down to 12 dof for the first 12 modes (i.e., a = 12 and m = 12). The active dof selected were 2X, 4X, 6X, 8Y, 10Y, 12Y, 14X, 16X, 18X, 20Y, 21Y, and 23Y as shown in Figure 2. Using this collection of dof, a reduced model eigensolution was performed.

The natural frequencies of the full 72 dof model and the reduced 12 dof model (Model 1) are shown in Table 1. From this table, it can be seen that the frequencies of the reduced system are the same as the frequencies of the full system for the modes under consideration. In addition, expanding the reduced model back to the full set of 72 dof results in mode shapes that are equal to those of the original analytical model, as expected.

The case when the number of active dof is equal to the number of modes preserved (a = m) is defined as the exact condition for SEREP. The reduced model obtained for this case is a truly an equivalent system.

Comparison of Frequencies
for Exact Model Reduction

Mode	Reference Model	Model 1
1	32.758	32.758
2	109.49	109.49
3	116.40	116.40
4	129.67	129.67
5	310.02	310.02
6	355.58	355.58
7	458.65	458.65
8	580.00	580.00
9	610.90	610.90
10	701.00	701.00
11	765.78	765.78
12	802.65	802.65

Model 1 - Modes 1,2,3,4,5,6,
7,8,9,10,11,12

ADOF 2X, 4X, 6X, 8Y, 10Y, 12Y,
14Y, 16X, 18X, 20Y, 21Y, 23Y

TABLE 1

Effects of Mode Selection

To examine the point that an arbitrary selection of modes can be preserved in the reduced model formulation, the basic FRAME structure was reduced using different collections of modes. A comparison of the actual full system's natural frequencies and the reduced system's natural frequencies was made.

The reduced system is comprised of the same 12 dof (i.e., a = 12) as used in the previous example. Three reduced models were examined. In these models, different sets of five modes (m = 5) are used in SEREP to produce the reduced 12 dof models.

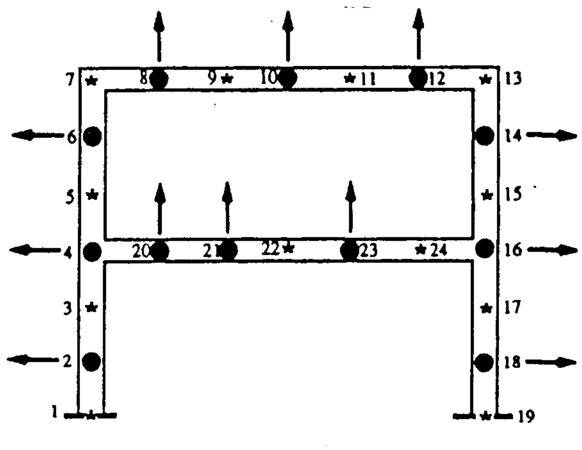
<u>Model 2</u> - Modes 1 to 5 from the basic FRAME model.

<u>Model 3</u> - Modes 2, 4, 6, 8, 10 from the basic FRAME model.

<u>Model 4</u> - Using modes 3, 4, 7, 9, 12 from the basic FRAME model.

Note, in these three cases, the number of dof in the reduced system models is greater than the number of modes that are used to develop the mapping matrix (i.e., a > m). The rank of the mass and stiffness matrices for the reduced system models equals 'm', the number of modes that are used to map the full system down to the reduced state. The order of the mass and stiffness matrices of the reduced system are of order 'a' (see Appendix A). Therefore, the mass and stiffness matrices are rank deficient.

Eigensolutions performed on the reduced 12 x 12 matrices for Models 2, 3 and 4 can only produce 5 eigenvalues since these system matrices are rank deficient and of rank 5. It can be seen from Table 2 that SEREP yields reduced matrices which preserve the arbitrarily selected modes used in the formulation of the $\underset{\sim}{T}_u$ matrix.



REDUCED 12 DEGREE OF FREEDOM
FRAME MODEL

FIGURE 2

Comparison of Frequencies
for Arbitrary Selection of Modes

Mode	Reference Model	Model 2	Model 3	Model 4
1	32.758	32.758	-	-
2	109.49	109.49	109.49	-
3	116.40	116.40	-	116.40
4	129.67	129.67	129.67	129.67
5	310.02	310.02	-	-
6	355.58	-	355.58	-
7	458.65	-	-	458.65
8	580.00	-	580.00	-
9	610.90	-	-	610.90
10	701.00	-	701.00	-
11	765.78	-	-	-
12	802.65	-	-	802.65

Model 2 - Modes 1,2,3,4,5
Model 3 - Modes 2,4,6,8,10
Model 4 - Modes 3,4,7,9,12

ADOF used in each of the above models:
2X,4X,6X,8Y,10Y,12Y,14Y,16X,18X,20Y,21Y,23Y.

TABLE 2

Effects of Point Selection

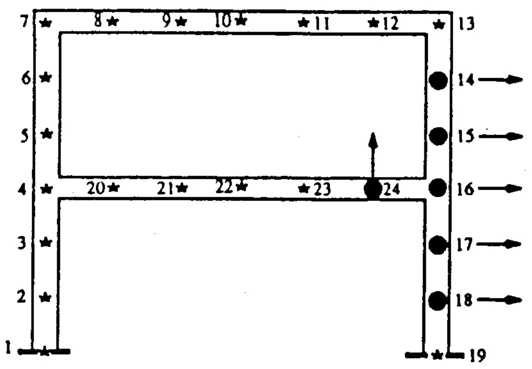
While it is true that the quality of results obtained using most reduction processes depends on the selection of reduced system dof, the same can not be said for SEREP.

To illustrate this point, two reduced system models are developed from the frame model using SEREP. The two models are comprised of an arbitrary selection of active dof as well as an arbitrary selection of modes.

<u>Model 5</u> - Modes 1, 3, 5 with active dofs 14X, 15X, 16X, 17X, 18X, 24Y as shown in Figure 3

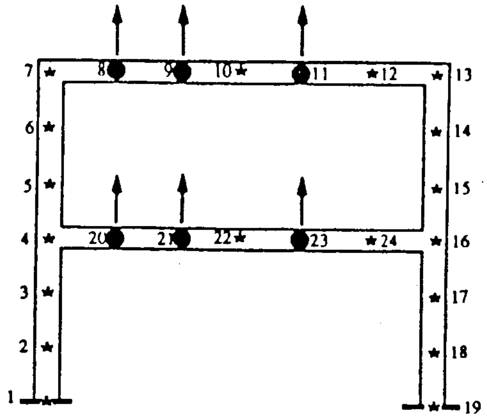
<u>Model 6</u> - Modes 2, 4, 5, 6 with active dofs 8Y, 9Y, 11Y, 20Y, 21Y, 23Y as shown in Figure 4

For other reduction techniques, the above selection of dof for a reduced system model would be considered poor. Table 3 lists the results of the frequencies of the reduced system model. It can be seen that even with an arbitrary selection of dof and for the specified modes to be tracked, SEREP produced reduced models which preserved all of the desired system dynamics at the selected dof in the reduced models.



REDUCED 6 DEGREE OF FREEDOM
FRAME MODEL
MODEL #5

FIGURE 3



REDUCED 6 DEGREE OF FREEDOM
FRAME MODEL
MODEL #6

FIGURE 4

Comparison of Frequencies
for Arbitrary Selection of DOF
and Arbitrary Selection of Modes

Mode	Reference Model	Model 5	Model 6
1	32.758	32.758	-
2	109.49	-	109.49
3	116.40	116.40	-
4	129.67	-	129.67
5	310.02	310.02	310.02
6	355.58	-	355.58
7	458.65		
8	580.00		
9	610.90		
10	701.00		
11	765.78		
12	802.65		

Model 5 - Modes : 1,3,5
ADOF : 14X,15X,16X,17X,18X,24Y

Model 6 - Modes : 2,4,5,6
ADOF : 8Y,9Y,11Y,20Y,21Y,23Y

TABLE 3

General Applications for SEREP

From the examples presented above, it can be seen that SEREP will produce a reduced model which preserves the desired system dynamics for an arbitrary selection of points and for an arbitrary selection of modes. These unique features inherent in the SEREP mapping process provide a tremendous capability for a wide assortment of structural dynamic modelling applications as well as general modal analysis applications such as

- Expansion of experimental modal data to include rotational dof as well as unmeasured translatory dof information needed for system modelling and structural dynamic modification studies (1)

- Correlation of analytical and experimental modal data and pseudo-orthogonality checks (2) can be performed at either the reduced state coresponding to the tested dof or at the full state corresponding to the full system model

- Reduction of large analytical models to a much smaller model used for the study of both linear and nonlinear response (3); in particular, nonlinear response can be performed on reduced models which provides a dramatic saving on computation time

- Improvement of analytical models using the measured modal data (4) can be made at either the test set of dof using the reduced mass and stiffness matrices or at the full set of dof using the expanded experimental modal vectors

SUMMARY

The System Equivalent Reduction Expansion Process (SEREP) is an exact method for mapping large analytical models down to much smaller equivalent reduced models. Originally developed for the purpose of generating rotational dof for experimental modal data, SEREP is being used in applications such as linear and nonlinear system response, orthogonality checks between analytical and test derived modal vectors, and analytical model improvement.

Unlike other reduction processes which develop reduced systems whose eigensolutions approximate the eigensolution of the full system model, SEREP develops reduced models whose frequencies and mode shapes are exactly the same as the full system model for the selected modes of interest.

Unlike other reduction tecniques, whose quality of results depends greatly on which of the full system's dof are selected to remain in the reduced model, the quality of results obtained from SEREP is insensitive to the selection of the full system dof that are to remain in the reduced system model.

Finally, SEREP can be used to track as many or as few of the full system modes as desired. This is not true of other reduction processes.

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APPENDIX A

This appendix is designed to supplement the equations and discussion of the accompanying paper. Elements of linear algebra such as generalized inverse, singular value decomposition, rank of reduced system matrices and the equivalence of the reduced eigensystem are presented.

Generalized Inverse

The inverse specification of a set of equations

$$\underline{X}_a = \underset{\sim}{U}_a \ \underline{P} \qquad (A.1)$$

where the number of unknowns is not equal to the number of equations requires a generalized inverse of the matrix equations. There are two possible solution types for this situation:

A) when the number of equations 'a' are greater than or equal to the number of solution variables 'm' (an over-specification or equivalence of the system)

and

B) when the number of equations are less than the number of solution variables (an under-specification of the system).

Type A: a > m

The system of Equations (A.1) can be put into normal form by projecting the system equations since the number of equations is greater than or equal to the number of unknowns as

$$\underline{Y}_m = \underset{\sim}{U}_a^T \ \underline{X}_a \qquad (A.2)$$

Applying this equation to Equation (A.1) produces

$$\underline{Y}_m = \underset{\sim}{U}_a^T \ \underset{\sim}{U}_a \ \underline{P}_m \qquad (A.3)$$

where $\underline{P}_m$ implies an approximate solution of $\underline{P}$. The square coefficient matrix of $\underline{P}_m$ will, in general, be fully ranked such that it possesses an inverse. The standard inverse of this matrix may pose some difficulty which will require the singular valued decomposition solution (to be discussed later). Symbolically, Equation (A.3) is used to solve for $\underline{P}_m$ as

$$\underline{P}_m = (\underset{\sim}{U}_a^T \ \underset{\sim}{U}_a)^{-1} \ \underline{Y}_m \qquad (A.4)$$

Substituting Equation (A.2) into (A.4) produces the general form of the solution as

$$\underline{P}_m = [(\underset{\sim}{U}_a^T \ \underset{\sim}{U}_a)^{-1} \ \underset{\sim}{U}_a^T] \ \underline{X}_a \qquad (A.5)$$

with the coefficient matrix in the square brackets being the generalized inverse of $\underset{\sim}{U}_a$ given in Equation (5) as

$$\underset{\sim}{U}_a^g = (\underset{\sim}{U}_a^T \ \underset{\sim}{U}_a)^{-1} \ \underset{\sim}{U}_a^T$$

Equation (A.5) represents the 'best' solution of the 'm' variables given the system of 'a' equations. It is important to note that Equation (A.5) can alternately be obtained by using a least squares solution if the minimization of the error squared between $\underline{X}_a$ (exact) and $\underset{\sim}{U}_a \underline{P}_m$ (approximate) is performed.

Type B: a < m

The solution of this condition is quite different since we have less equations than solution unknowns. The best that this solution can be is an average solution of the equations in the system. If system matrices are formed for this condition, it can be shown that the system matrices are rank 'a' and are therefore not rank deficient. But the system modal variables 'm' have been 'meld' into 'a' equations thus producing an average of the system variables.

Developing the solution for this condition requires that a set of 'a' variables be projected to the 'm' system variables as

$$\underline{P}_m = \underset{\sim}{U}_a^T \underline{P}_a \tag{A.6}$$

where $\underline{P}_a$ is the approximate solution set.

The opposite of this statement implies that the 'm' system variables will be averaged into the 'a' system of equations.

Substituting into Equation (A.1) produces

$$\underline{X}_a = (\underset{\sim}{U}_a \underset{\sim}{U}_a^T) \underline{P}_a \tag{A.7}$$

which, in turn, can be used to solve for the $\underline{P}_a$ variables. The coefficient matrix in Equation (A.7) can be shown to be of proper rank to possess a standard inverse such that

$$\underline{P}_a = (\underset{\sim}{U}_a \underset{\sim}{U}_a^T)^{-1} \underline{X}_a \tag{A.8}$$

Using Equation (A.8) in (A.6) produces the general form of the average solution where a<m as

$$\underline{P}_m = [\underset{\sim}{U}_a^T (\underset{\sim}{U}_a \underset{\sim}{U}_a^T)^{-1}] \underline{X}_a \tag{A.9}$$

where the coefficient matrix in the square brackets being the generalized inverse of $\underset{\sim}{U}_a$ given in Equation (7) as

$$\underset{\sim}{U}_a^g = \underset{\sim}{U}_a^T (\underset{\sim}{U}_a \underset{\sim}{U}_a^T)^{-1}$$

Equation (A.9) represents an average solution of an under-specified set of equations containing 'm' system variables.

Singular Valued Decomposition

Probably the most powerful technique in linear algebra and numerical methods is the Singular Valued Decomposition (SVD) of a matrix. As stated above, the coefficient matrices can be inverted safely using the SVD process. The procedure is also used to determine the rank of a matrix since the number of singular values (that is, those values that are above a threshold value of the machine zero) represents the rank of the matrix. The complete development of the SVD process is given in reference (5) which can be simply stated in the following section.

Any matrix $\underset{\sim}{A}$ of order (n x m) can be decomposed into its orthonormal matrices and singular values as

$$\underset{\sim}{A} = \underset{\sim}{L} \underset{\sim}{S} \underset{\sim}{R}^T \tag{A.10}$$

where

$\underset{\sim}{L}$ is an orthonormal matrix of order (n x n),
$\underset{\sim}{R}$ is an orthonormal matrix of order (m x m),
$\underset{\sim}{S}$ is the matrix of order (n x m) containing the singular values as

$$\underset{\sim}{S} = \begin{bmatrix} \diagdown\underset{\sim}{\sigma}_r\diagdown & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \tag{A.11}$$

The $\underset{\sim}{S}$ matrix has a special form where the upper left partition contains $\underset{\sim}{\sigma}_r$ which is a diagonal matrix of order (r x r) containing the singular values of matrix $\underset{\sim}{A}$. The order of matrix $\underset{\sim}{\sigma}_r$ is the rank of the matrix $\underset{\sim}{A}$.

Reduced System Matrices

The reduced system matrices $\underset{\sim}{M}_a$ and $\underset{\sim}{K}_a$ in Equations (12) and (13) have been formed by transforming the full system variable $\underline{X}_n$ into the reduced system variable $\underline{X}_a$ using Equation (11). At the same time the system equations are placed into normal form by projecting the equation set using the $\underset{\sim}{T}_u$ transpose operator. The resulting reduced mass matrix is

$$\underset{\sim}{M}_a = \underset{\sim}{U}_a^{gT} (\underset{\sim}{U}_n^T \underset{\sim}{M}_n \underset{\sim}{U}_n) \underset{\sim}{U}_a^g \tag{A.12}$$

$$\underset{\sim}{M}_a = \underset{\sim}{U}_a^{gT} \underset{\sim}{U}_a^g$$

since the original modal vectors were normalized to unit modal mass. In a similar fashion, the stiffness matrix is

$$\underset{\sim}{K}_a = \underset{\sim}{U}_a^{gT} \diagdown\underset{\sim}{\omega}^2\diagdown \underset{\sim}{U}_a^g \tag{A.13}$$

where $\diagdown\underset{\sim}{\omega}^2\diagdown$ is the modal stiffness matrix of the original system.

Determining the rank of these matrices, assuming that a > m, requires the knowledge of the rank of $\underset{\sim}{U}_a$ which is used to form $\underset{\sim}{U}_a^g$. Using SVD, $\underset{\sim}{U}_a$ can be decomposed into

$$\underset{\sim}{U}_a = \underset{\sim}{L} \underset{\sim}{S} \underset{\sim}{R}^T \tag{A.14}$$

where

$$\underset{\sim}{S} = \begin{bmatrix} \diagdown\underset{\sim}{\sigma}_m\diagdown \\ \underset{\sim}{0} \end{bmatrix} \tag{A.15}$$

and is of rank 'm'. $\underset{\sim}{S}$ is of this form and rank because the $\underset{\sim}{U}_a$ matrix was partitioned from the $\underset{\sim}{U}_n$ matrix which contains 'm' linearly independent vectors by column.

The $\underset{\sim}{U}_a^g$ matrix is then formed using Equation (5) as

$$\underset{\sim}{U}_a^g = \underset{\sim}{R}\ \underset{\sim}{S}^g\ \underset{\sim}{L}^T \tag{A.16}$$

where

$$\underset{\sim}{S}^g = [\ \underset{\sim}{\diagdown \sigma_m^{-1}\diagdown}\quad \underset{\sim}{0}\ ] \tag{A.17}$$

and noting the orthonormal conditions of L and R. Equations (A.16) and (A.17) define the SVD of the U_a^g matrix which implies that its rank is also 'm'.

Substituting Equation (A.16) and (A.17) into the reduced system matrices of Equation (A.12) and (A.13) produces

$$\underset{\sim}{M}_a = \underset{\sim}{L}\ [\ \underset{\sim}{S}_g^T\ \underset{\sim}{S}_g\ ]\ \underset{\sim}{L}^T \tag{A.18}$$

where

$$\underset{\sim}{S}^{gT}\ \underset{\sim}{S}^g = \begin{bmatrix} \underset{\sim}{\diagdown \sigma_m^{-2}\diagdown} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \tag{A.19}$$

From Equation (A.18) and (A.19), it has been shown that the rank of $\underset{\sim}{M}_a$ is rank 'm' which is less than the order of the matrix; therefore, the reduced system mass matrix is rank deficient.

In a similar fashion, using the definitions of $\underset{\sim}{U}_a^g$ in the reduced system stiffness matrix in Equation (A.13) produces

$$\underset{\sim}{K}_a = \underset{\sim}{L}\ [\ \underset{\sim}{S}^{gT}\ \underset{\sim}{R}^T\ \underset{\sim}{\diagdown \omega^2\diagdown}\ \underset{\sim}{R}\ \underset{\sim}{S}^g\ ]\ \underset{\sim}{L}^T \tag{A.20}$$

where the terms within the square brackets can be reduced using SVD to

$$\underset{\sim}{S}^{gT}\ \underset{\sim}{R}^T\ \underset{\sim}{\diagdown \omega^2\diagdown}\ \underset{\sim}{R}\ \underset{\sim}{S}^g = \underset{\sim}{L}_1\ \underset{\sim}{S}_1\ \underset{\sim}{L}_1^T \tag{A.21}$$

where

$$\underset{\sim}{S}_1 = \begin{bmatrix} \underset{\sim}{\diagdown \gamma_m\diagdown} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} \tag{A.22}$$

Thus, the reduced stiffness matrix is also rank deficient at rank 'm'.

The SEREP condition occurs when 'a' is equal to 'm'. It is for this condition that the reduced system is truly equivalent and the rank is equal to the order of the reduced matrices. As noted previously, the case when 'a' is less than 'm', the system matrices can be shown to be of proper rank but this condition is not normally useful since the solution is only an average. It will be shown in the next section that, even though the reduced system matrices, $\underset{\sim}{M}_a$ and $\underset{\sim}{K}_a$, are rank deficient ('a' greater than 'm'), the reduced eigen-solution will produce the proper eigen-system.

<u>Reduced Eigen-Solution</u>

The eigensolution of the reduced system matrices is obtained from Equation (14) as

$$[\ \underset{\sim}{K}_a - \lambda\ \underset{\sim}{M}_a\ ]\ \underset{\sim}{X}_a = \underset{\sim}{0} \tag{A.23}$$

Substituting Equation (A.18), (A.20) and (A.17) into the above equation produces

$$\underset{\sim}{L}\begin{bmatrix} (\underset{\sim}{\sigma}_m^{-1}\ \underset{\sim}{R}^T \underset{\sim}{\diagdown \omega^2\diagdown}\ \underset{\sim}{R}\ \underset{\sim}{\sigma}_m^{-1} - \lambda\ \underset{\sim}{\sigma}_m^{-2}) & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{0} - \lambda\ \underset{\sim}{0}) \end{bmatrix}\underset{\sim}{L}^T\underset{\sim}{X}_a = \underset{\sim}{0} \tag{A.24}$$

which can be further reduced to

$$[\ \underset{\sim}{U}_a^{gT}\quad \underset{\sim}{0}\ ]\begin{bmatrix} (\underset{\sim}{\diagdown \omega^2\diagdown} - \lambda \underset{\sim}{I}) & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{0} - \lambda\ \underset{\sim}{0}) \end{bmatrix}\begin{bmatrix} \underset{\sim}{U}_a^g \\ \underset{\sim}{0} \end{bmatrix}\underset{\sim}{X}_a = \underset{\sim}{0} \tag{A.25}$$

and finally down to

$$\underset{\sim}{U}_a^{gT}\ [\ \underset{\sim}{\diagdown \omega^2\diagdown} - \lambda\ \underset{\sim}{I}\ ]\ \underset{\sim}{U}_a^g\ \underset{\sim}{X}_a = \underset{\sim}{0} \tag{A.26}$$

or

$$[\ \underset{\sim}{\omega}^2 - \lambda\ \underset{\sim}{I}\ ]\ \underset{\sim}{P} = \underset{\sim}{0} \tag{A.27}$$

using Equation (6).

It is shown in Equations (A.24) and (A.25) that the reduced system eigensolution contains two groups of eigenvalues. The first group defined in the upper partition is the group of eigenvalues of the original system that was used in the reduced system definition of modal frequencies. The second group results from the fact that the system matrices are rank deficient and correspond to a zero modal mass and zero modal stiffness and therefore are indeterminant. These values are considered null values of the eigensystem.

The final Equation (A.27) indicates that the reduced system eigensolution produces the same eigenvalues and vectors of the original system once the null values have been removed.