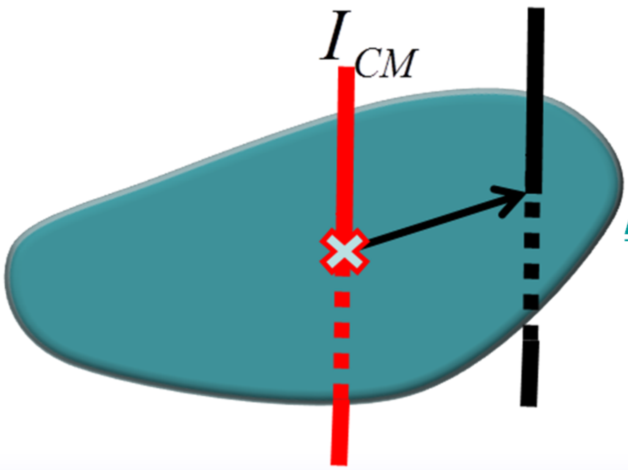
Lecture 18

Physics I
11.18.2013

Chapter 10

Rotational Dynamics
Moment of Inertia.



Course website:
http://faculty.uml.edu/Andriy_Danylov/Teaching/PhysicsI

Lecture Capture:
http://echo360.uml.edu/danylov2013/physics1fall.html

Outline

Chapter 10

- Moment of Inertia
- Parallel Axis Theorem
- Rotational kinetic energy
- Rolling

Newton's 2nd law of rotation

Force causes linear acceleration: (N.2nd law):

$$\vec{F} = m\vec{a}$$

Torque causes angular acceleration:

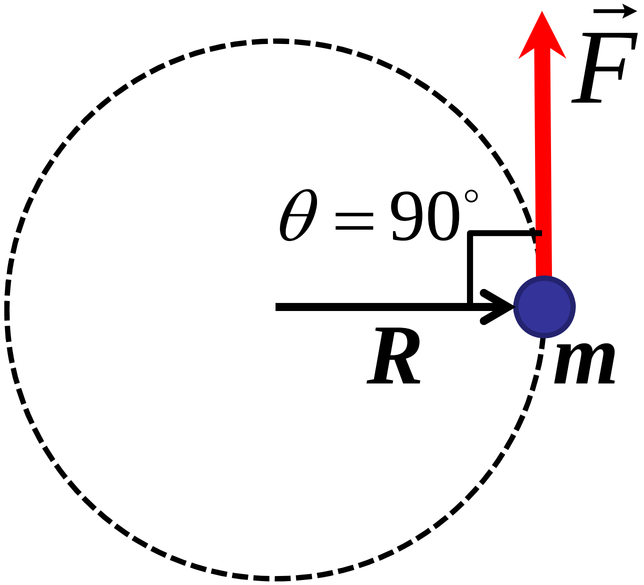
$$\boxed{\vec{\tau} = I\vec{\alpha}}$$

I is the Moment of Inertia (rotational equivalent of mass)

Moment of inertia of a single particle

A point mass is located at a distance R from an axis of rotation.
A force is applied perpendicular to R.

Let's find a relation between torque and angular acceleration:



By definition: $\tau = RF\sin\theta = RF$

N. 2nd law: $F = ma = mR\alpha$

Recall, last class: $a = R\alpha$

As a result, torque is:

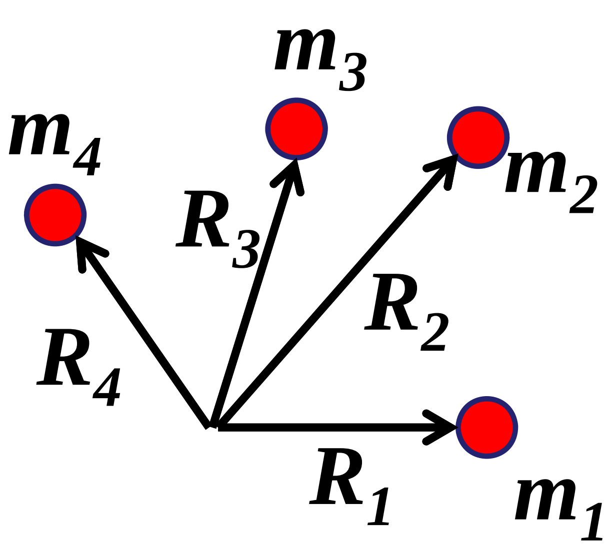
$$\tau = R(mR\alpha) = (mR^2)\alpha$$

Moment of inertia of a single particle: $I = mR^2$

Rotational N. 2nd law: $\tau = I\alpha$

Moment of inertia of many particle

If we have many point masses m_i, located at distances R_i from an axis of rotation. A force is applied perpendicular to R.



Moment of inertia of N masses:

$$I = m_1 R_1^2 + m_2 R_2^2 + m_3 R_3^2 + \ldots$$

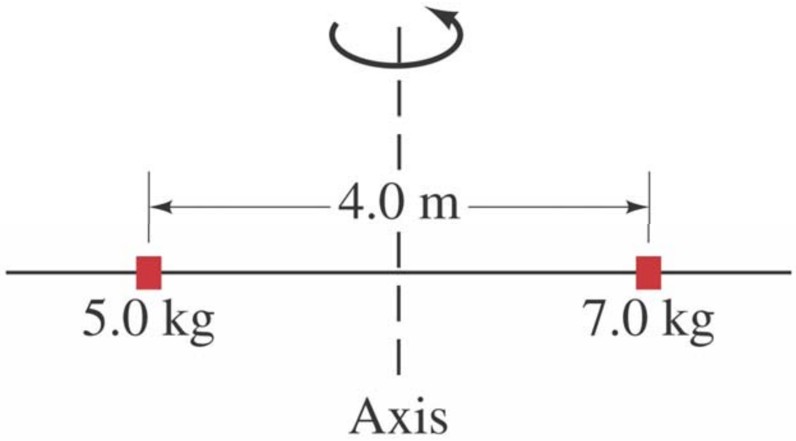
$$I = \sum_{i=1}^{N} m_i R_i^2$$

Rotational N. 2nd law: $\tau = I\alpha$

$$\tau = \left(\sum_{i=1}^{N} m_i R_i^2\right)\alpha$$

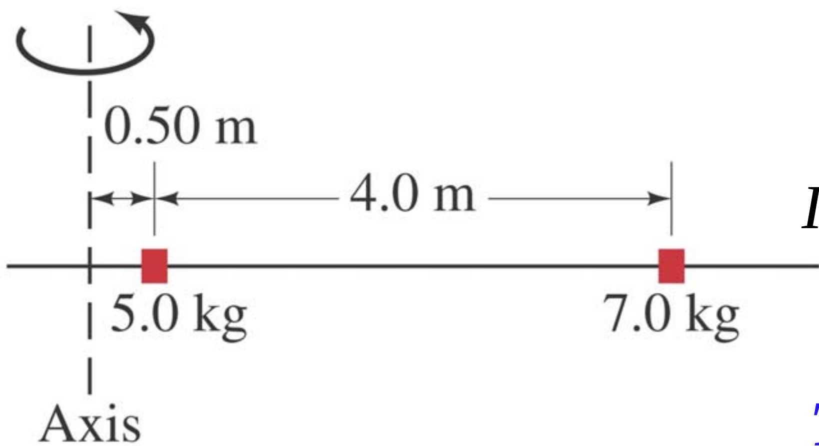
Example: Moments of Inertia of two points

Two point masses connected to a massless rod



$$I = m_1 R_1^2 + m_2 R_2^2$$

$$I = 5kg(2m)^2 + 7kg(2m)^2 = 48\,kg \cdot m^2$$



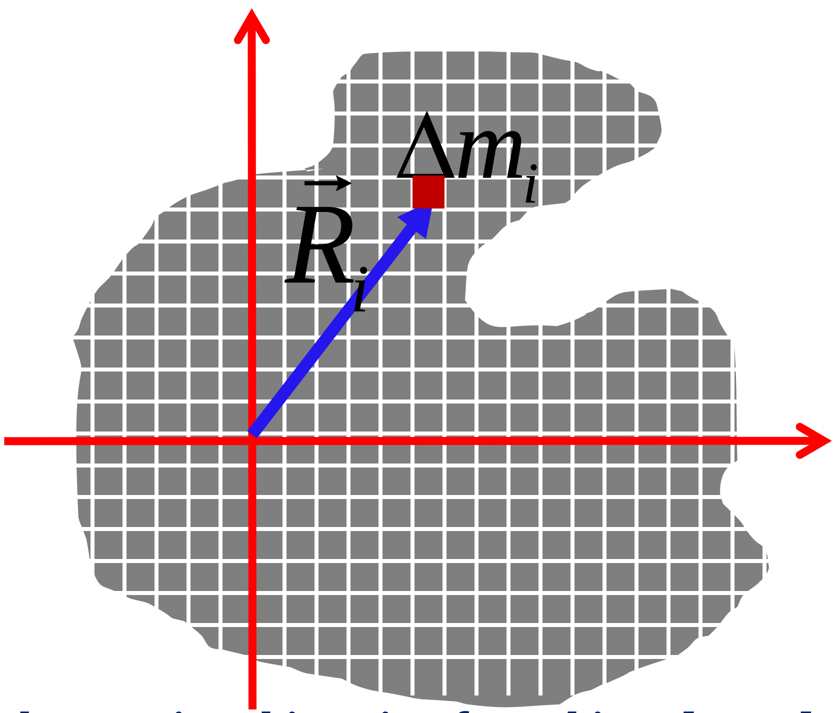
$$I = m_1 R_1^2 + m_2 R_2^2$$

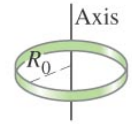
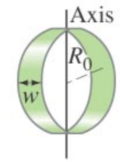
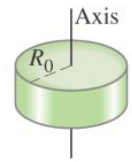
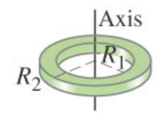
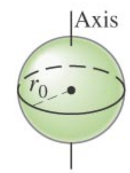
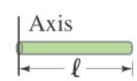
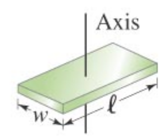
$$I = 5kg(0.5m)^2 + 7kg(4.5m)^2 = 143\,kg \cdot m^2$$

The distribution of mass matters

Moment of inertia for extended objects

$$I = \int R^2 dm$$

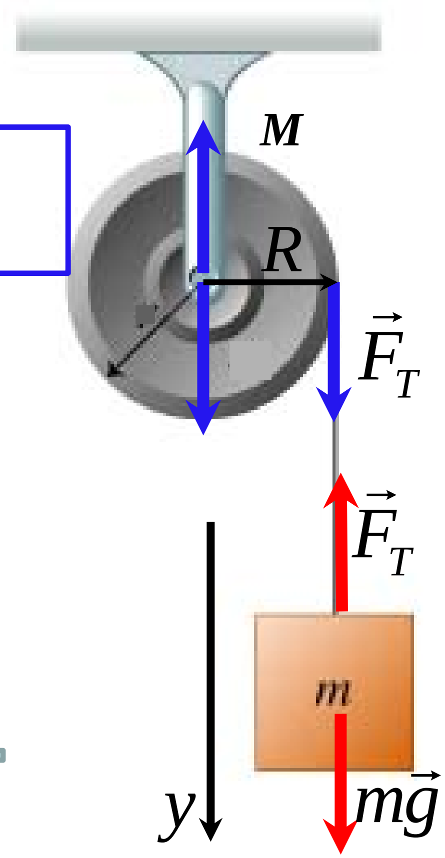


	Object	Location of axis		Moment of inertia
(a)	**Thin hoop,** radius R_0	Through center		MR_0^2
(b)	**Thin hoop,** radius R_0 width w	Through central diameter		$\frac{1}{2}MR_0^2 + \frac{1}{12}Mw^2$
(c)	**Solid cylinder,** radius R_0	Through center		$\frac{1}{2}MR_0^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center		$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius r_0	Through center		$\frac{2}{5}Mr_0^2$
(f)	**Long uniform rod,** length ℓ	Through center		$\frac{1}{12}M\ell^2$
(g)	**Long uniform rod,** length ℓ	Through end		$\frac{1}{3}M\ell^2$
(h)	**Rectangular thin plate,** length ℓ, width w	Through center		$\frac{1}{12}M(\ell^2 + w^2)$

The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation— compare (f) and (g), for example.

Example: pulley and mass/Physics

An object of mass m is hung from a cylindrical pulley of radius R and mass M and released from rest. What is the acceleration of the object?



We have two objects in the system:

- translational motion of m, described by N.2nd law

$$(1) \qquad \sum \vec{F} = m\vec{a}$$

- rotational motion of M, described by the rotational N.2nd law

$$(2) \qquad \sum \vec{\tau} = I\vec{\alpha}$$

- and there is a useful eq-n, which links these 2 eq-ns:

$$(3) \qquad a_{\tan} = R\alpha = a$$

$$(1) \ \sum \vec{F} = m\vec{a} \Rightarrow mg - F_T = ma$$

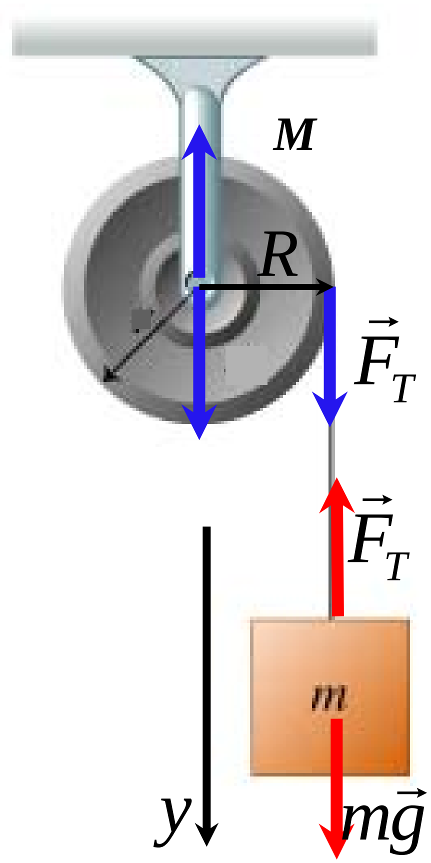
$$(2) \ \sum \vec{\tau} = I\vec{\alpha} \Rightarrow F_T R \cancel{\sin 90^\circ}^{\,1} = I\alpha$$

$$(3) \ a_{\tan} = R\alpha = a \Rightarrow \alpha = a/R$$

Physics is over.
Now, it is pure Algebra.
3 eq-ns and 3 unknowns

$$I = (see\ table 10-20)$$

Example: pulley and mass/Algebra



$$(1) \quad mg - F_T = ma$$
$$(2) \quad F_T R = I\alpha$$
$$(3) \quad \alpha = a/R$$

$$\begin{cases} mg - F_T = ma & \big| \times R \\ F_T R = I\dfrac{a}{R} & + \end{cases}$$

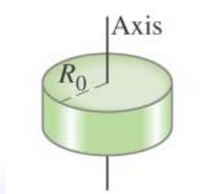
Multiply both sides of (1) by R and add (1) and (2) up: (F_T disappears)

$$mgR - F_T R + F_T R = maR + I\frac{a}{R}$$

$$a = \frac{mRg}{mR + \dfrac{I}{R}}$$

Or better

$$a = \frac{mg}{m + \dfrac{I}{R^2}}$$



$$I = (\text{see table } 10-20)$$

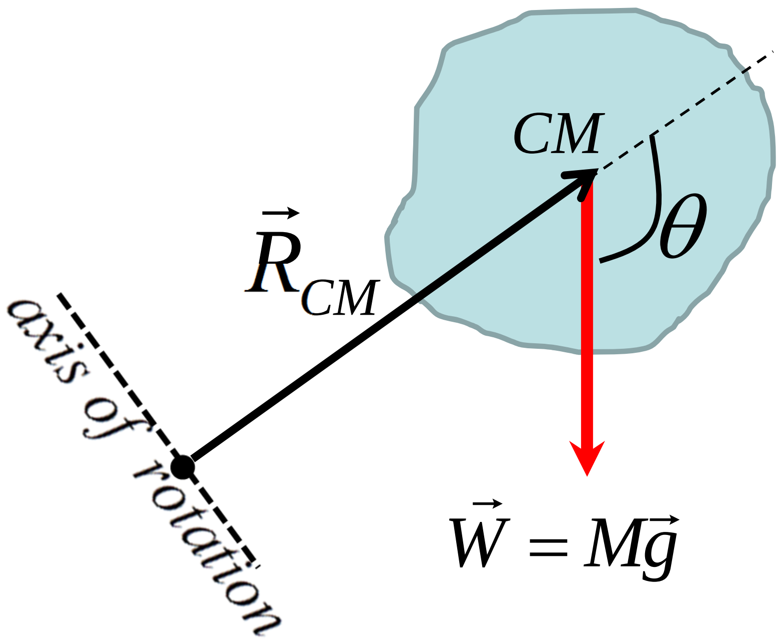
since $I = \dfrac{1}{2}MR^2$

$$a = \frac{mg}{m + \dfrac{M}{2}}$$

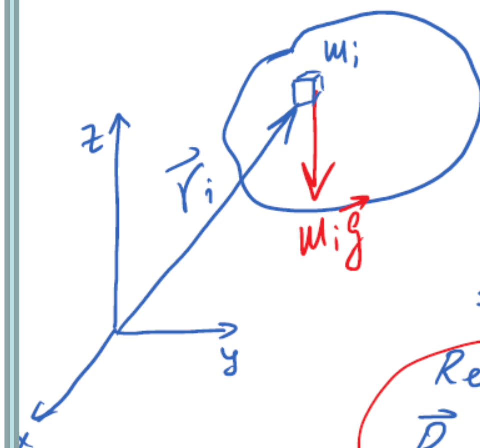
Torque due to gravity

- ***We often encounter systems in which there is a torque exerted by gravity. The torque on a body about any axis of rotation is***

$$\tau = R_{CM} W Sin\theta = R_{CM} Mg Sin\theta$$



The proof



$\vec{\tau}_i = \vec{r}_i \times m_i \vec{g}$

the total torque:

$\vec{\tau} = \sum \vec{\tau}_i = \sum_i \vec{r}_i \times m_i \vec{g} = \sum_i (m_i \vec{r}_i) \times \vec{g}$

Recall, CM definition:

$\vec{R}_{CM} = \frac{\sum m_i \vec{r}_i}{M}$; M – total mass

so $M\vec{R}_{cm} = \sum_i m_i \vec{r}_i$

Thus,

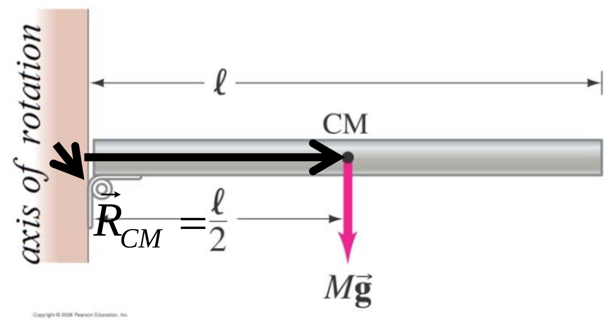
$\vec{\tau} = M\vec{R}_{cm} \times \vec{g} = \vec{R}_{cm} \times (M\vec{g}) = \vec{R}_{cm} \times \vec{W} = \vec{\tau}$ ∴

Example Problem: Falling rod

What is the angular acceleration of the rod shown below, if it is released from rest, at the moment it is released? What is the linear acceleration of the tip?

Rotational motion of the rod is described by the rotational N.2nd law

$$\sum \vec{\tau} = I\vec{\alpha}$$



Torque due to gravity (previous slide):

$$\tau = R_{CM} W Sin\theta = R_{CM} Mg Sin\theta$$

$$\tau = MgR_{CM} Sin90^\circ = Mg\, l/2$$

$$Mg\frac{l}{2} = I\alpha \Rightarrow \alpha = \frac{Mgl}{2I} = \frac{Mgl}{2(\frac{1}{3}Ml^2)} = \frac{3g}{2l}$$

$$I = \tfrac{1}{3}Ml^2$$

Each point on a rotating rigid body has the same angular acceleration (previous class)!

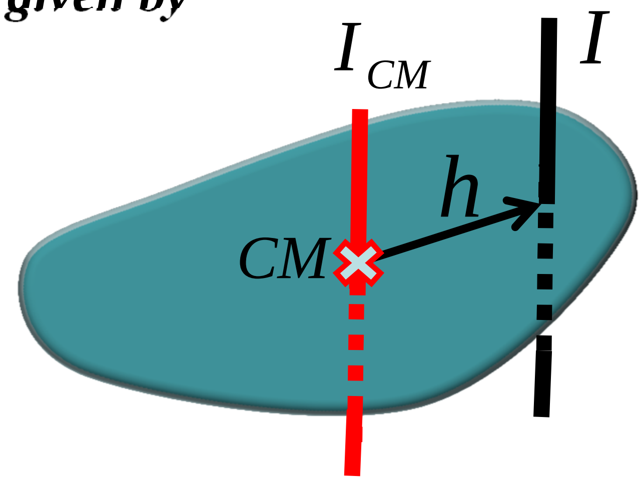
So we can apply it to the tip!!

$$a_{\tan} = R\alpha = l(3g/2l) = \frac{3g}{2}$$

Parallel Axis Theorem (w/out proof)

The moment of inertia about any axis parallel to that axis through the center of mass is given by

$$I = I_{CM} + Mh^2$$



I : ***moment of inertia about any parallel axis***

I_{CM} : moment of inertia about an axis through its center of mass

M : ***total mass***

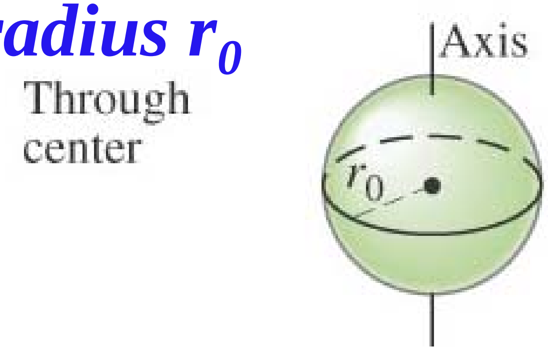
h : ***distance from a parallel axis to the center of mass.***

BTW: The moment of inertia of any object about an axis through its center of mass is the minimum moment of inertia for an axis in that direction in space.

Parallel Axis Theorem: Example/Sphere

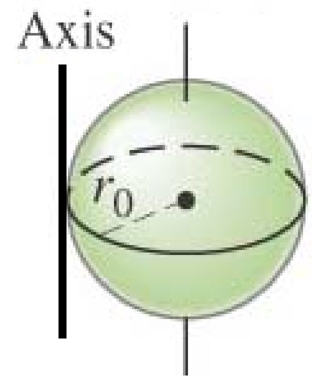
For a uniform sphere of radius r_0

Moment of inertia for the sphere, rotating about an axis through its center of mass



$$I_{CM} = \tfrac{2}{5} M r_0^2$$

Moment of inertia for the sphere about an axis going through the edge of the sphere?



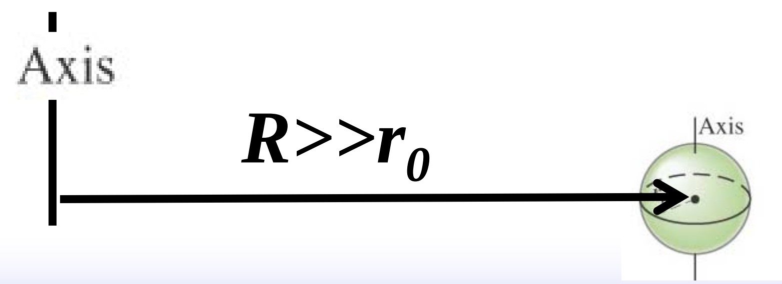
Apply Parallel Axis Theorem:

$$I = I_{CM} + Mh^2 = \tfrac{2}{5} M r_0^2 + M r_0^2 = \tfrac{7}{5} M r_0^2$$

Through an axis a distance $R \gg r_0$ from the center?

Very small

$$I = I_{CM} + Mh^2 = \tfrac{2}{5} M r_0^2 + MR^2 \approx MR^2$$

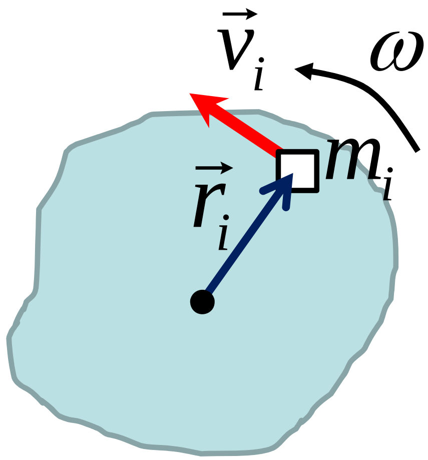


So, in this case we got a Moment of inertia of a single particle

Department of Physics and Applied Physics

Rotational Kinetic Energy

Simple derivation: for pure rotation



$$K_{rot} = \sum \tfrac{1}{2} m_i v_i^2 \qquad v_i = r_i \omega$$

$$K_{rot} = \sum \tfrac{1}{2} m_i (\omega r_i)^2 = \sum \tfrac{1}{2} m_i \omega^2 r_i^2$$

$$K_{rot} = \tfrac{1}{2} \omega^2 \sum m_i r_i^2$$

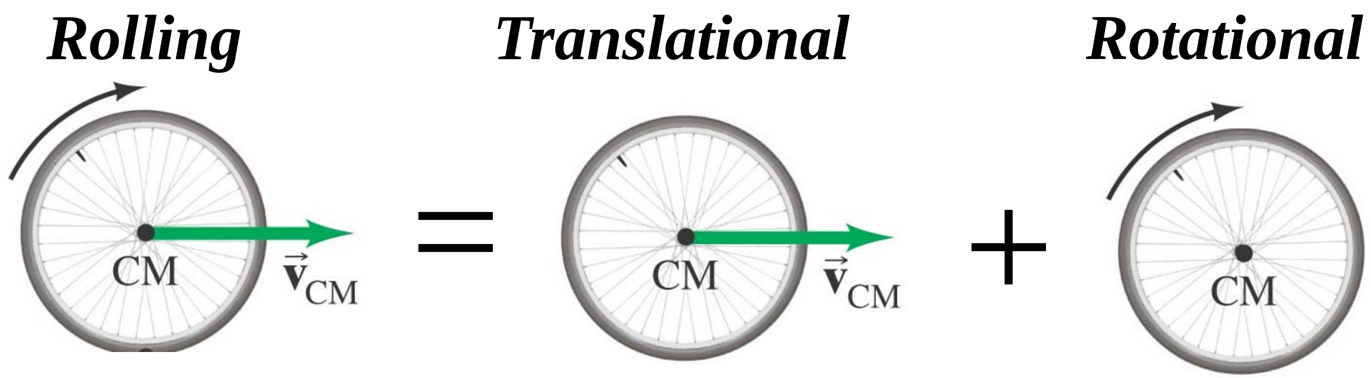
Since $I = \sum m_i r_i^2$

therefore $\boxed{K_{rot} = \tfrac{1}{2} I \omega^2}$

$K_{trans} = \dfrac{1}{2} m v^2$ ***Similar to linear motion*** $K_{rot} = \dfrac{1}{2} I \omega^2$

Total kinetic energy

Rolling motion can be resolved into two motions



$$Total\ Kinetic\ Energy = K_{CM} + K_{rot}$$

$$K_{tot} = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2$$

If there is no slipping ⇨ $v_{CM} = R\omega$

$$K_{tot} = \tfrac{1}{2} M R^2 \omega^2 + \tfrac{1}{2} I_{CM} \omega^2 \quad \Rightarrow \quad K_{tot} = \tfrac{1}{2} (M R^2 + I_{CM}) \omega^2$$

ConcepTest 1 Dumbbell I

A force is applied to a dumbbell for a certain period of time, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater center-of-mass speed ?

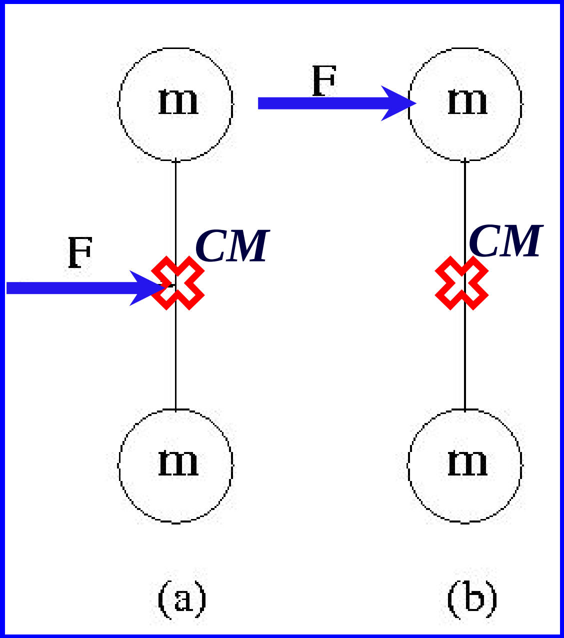
A) case (a)

B) case (b)

C) no difference

D) it depends on the rotational inertia of the dumbbell

Because the same force acts for the same time interval in both cases, the change in momentum must be the same, thus the CM velocity must be the same.



ConcepTest 2 Dumbbell II

A force is applied to a dumbbell for a certain period of time, first as in (a) and then as in (b). In which case does the dumbbell acquire the greater energy ?

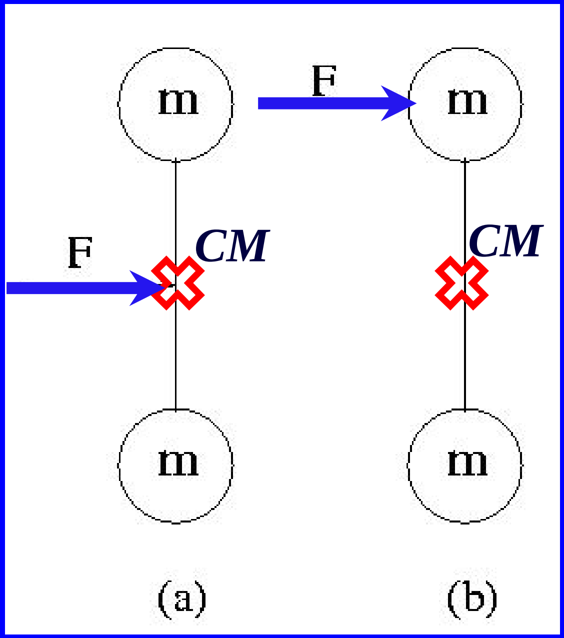
A) case (a)

B) case (b)

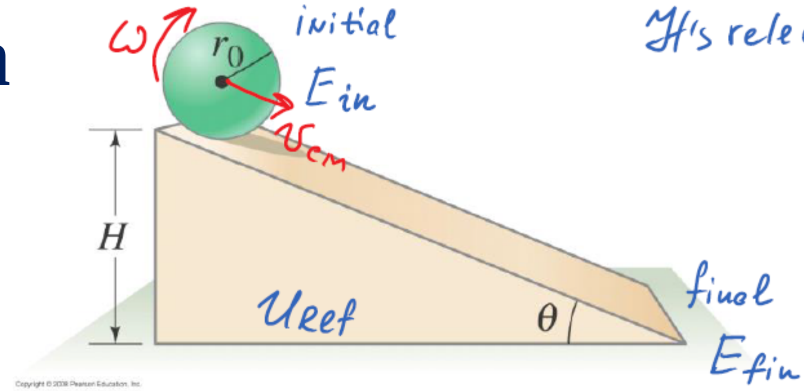
C) no difference

D) it depends on the rotational inertia of the dumbbell

If the CM velocities are the same, the translational kinetic energies must be the same. Because dumbbell (b) is also rotating, it has rotational kinetic energy in addition.



Sphere rolling down an incline



It's released from rest

Use conserv. of mech. energy: $E_{in} = E_{fin}$

$E_{in} = K_{tr}^{\,0} + K_{rot}^{\,0} + U_i = mgH$

$E_{fin} = K_{tr} + K_{rot} + U_f^{\,0\,(\text{ref. level})} = \frac{1}{2} m V_{CM}^2 + \frac{1}{2} I_{CM} \cdot \omega^2$

so, $\frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \omega^2 = mgH$

Since there is no slipping $V_{CM} = \omega \cdot R \Rightarrow \omega = \frac{V_{CM}}{R}$

$\frac{1}{2} m V_{CM}^2 + \frac{1}{2} I \cdot \frac{V_{CM}^2}{R^2} = mgH$; $I = \frac{2}{5} m \cdot R^2$ (table 10-20)

$\frac{V_{CM}^2}{2} \cdot \left(m + \frac{2}{5} m R^2 \cdot \frac{1}{R^2}\right) = \frac{7}{10} \cdot m \cdot V_{CM}^2 = mgH$, so

$$V_{CM} = \sqrt{\frac{10}{7} g H}$$

Compare: with

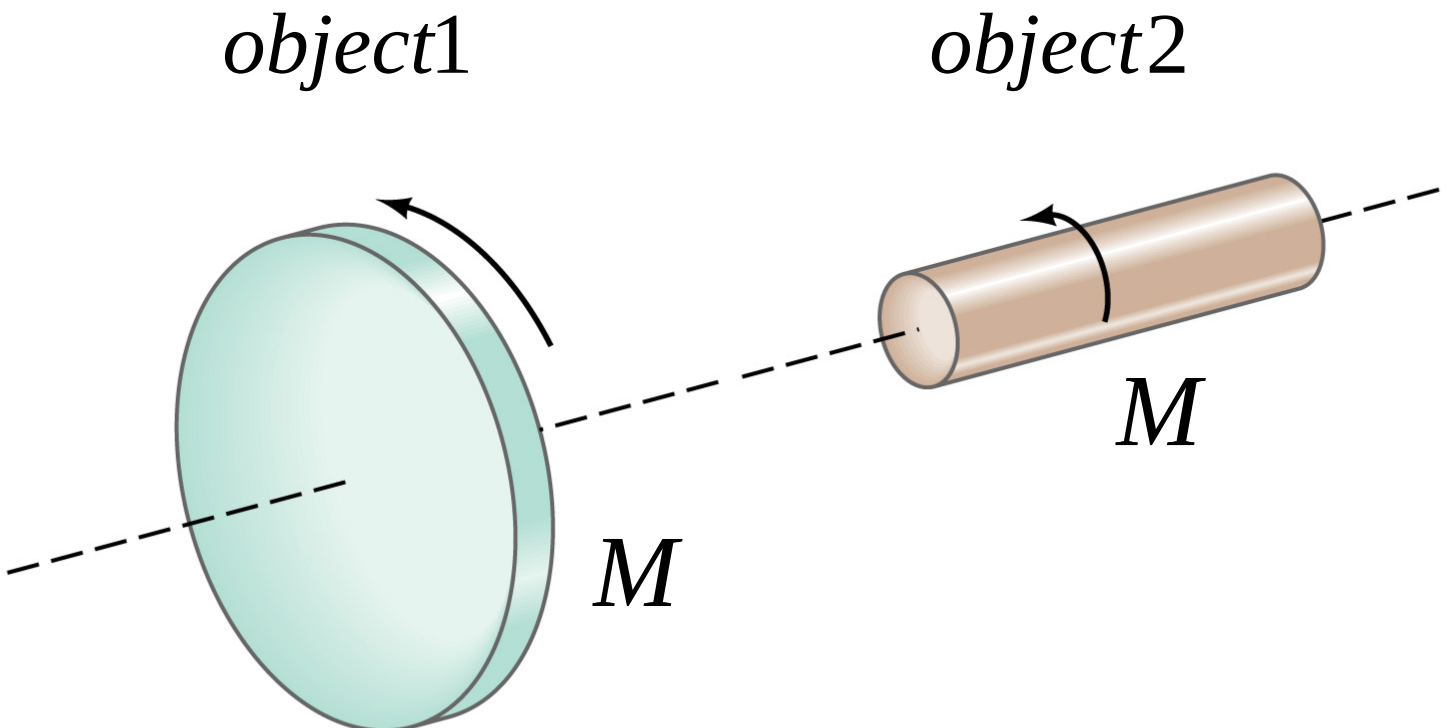
$V = \sqrt{2gH}$

Thank you

See you on Wednesday

Moment of inertia

The distribution of mass matters here—these two objects have the same mass, but the one on the left has a greater rotational inertia, as so much of its mass is far from the axis of rotation.



$$I_1 > I_2$$