Lecture 19

University of Massachusetts
UMASS Lowell
Physics I

Chapter 12

Center of Mass
Moment of Inertia

Course website:
http://faculty.uml.edu/Andriy_Danylov/Teaching/PhysicsI

Today we are going to discuss:

Chapter 12:

- ***Rotation about the Center of Mass:*** *Section 12.2*

  (skip "Finding the CM by Integration)

- ***Rotational Kinetic Energy:*** *Section 12.3*
- ***Moment of Inertia:*** *Section 12.4*

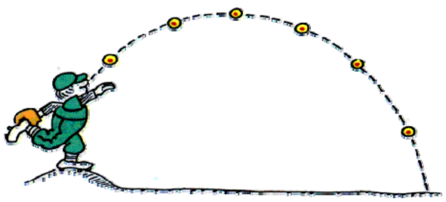
Center of Mass (CM)

Center of Mass (CM) idea

Center of Mass (CM)

We know how to address these problems:

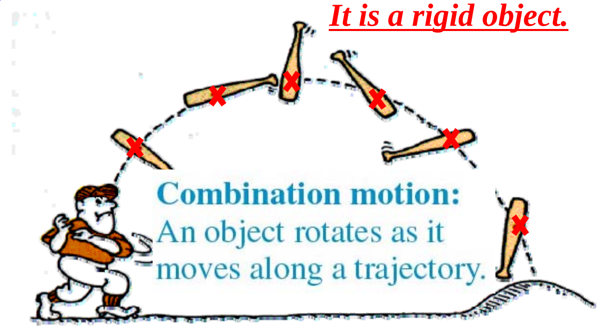
Translational motion:
The object as a whole moves along a trajectory but does not rotate.



Rotational motion:
The object rotates about a fixed point. Every point on the object moves in a circle.

We also know how to address this motion of a single particle - kinematic equations

How to describe motions like these?
It is a rigid object.



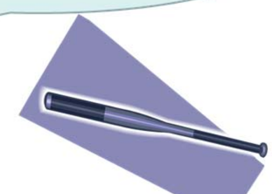
Combination motion:
An object rotates as it moves along a trajectory.

Translational plus rotational motion

*The general motion of an object can be considered as the sum of translational motion of a **certain point**, plus rotational motion about that point.*

That point is called the center of mass point.

How to find the center of mass?



A CM point depends only on the mass distribution of an object.

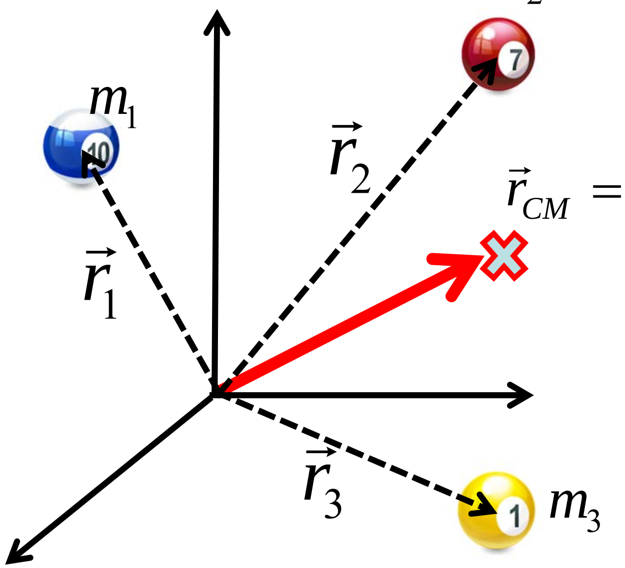
Center of Mass: *Definition*

Position vector of the CM: $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$

$M = m_1 + m_2 + m_3$
total mass of the system

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

The center of mass is the mass-weighted center of the object



$\vec{r}_{CM} = (x_{CM}, y_{CM}, z_{CM})$

Component form:

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

$$y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

$$z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

Example: Center of Mass Position (2 particles)

What is the center of mass of 2 point masses (m_A=1 kg and m_B=3 kg), at two different points: A=(0,0) and B=(2,4)?

By definition:

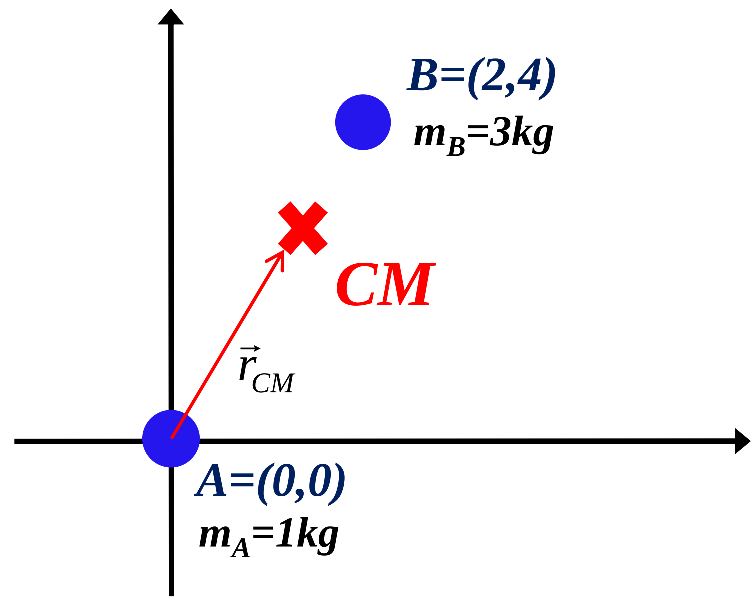
$$x_{CM} = \frac{m_A x_A + m_B x_B}{(m_A + m_B)}$$

$$y_{CM} = \frac{m_A y_A + m_B y_B}{(m_A + m_B)}$$

m_A=1 kg and m_B=3 kg

$$x_{CM} = \frac{(1\times 0)+(3\times 2)}{1+3} = 1.5$$

$$y_{CM} = \frac{(1\times 0)+(3\times 4)}{1+3} = 3$$

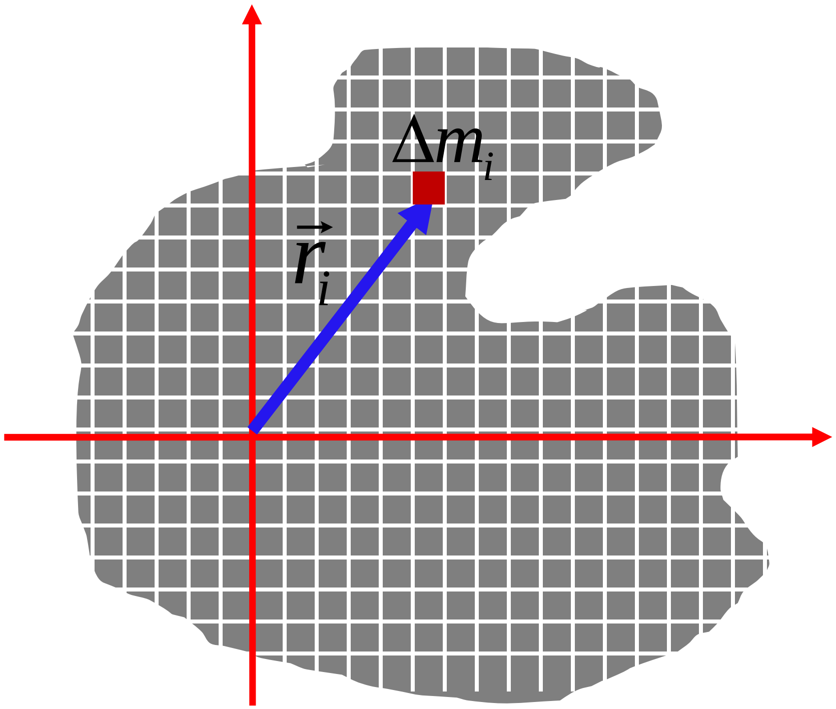


Or in a vector form: $\vec{r}_{CM} = 1.5\hat{i} + 3\hat{j}$

CM of a solid object

(But we are not going to use this since I promised to avoid integration)

Let's find CM of an extended body:



Before, for many particles we had $\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$

Now, let's divide extended mass into smaller sections Δm_i

$$\vec{r}_{CM} = \frac{1}{M}\sum_i \Delta m_i \vec{r}_i$$

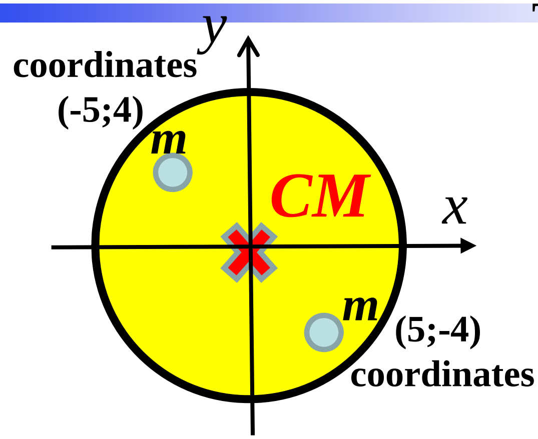
$$\vec{r}_{CM} = \frac{1}{M}\lim_{\Delta m_i \to 0}\sum_i \Delta m_i \vec{r}_i$$

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

$$x_{CM} = \frac{1}{M}\int x\,dm \qquad y_{CM} = \frac{1}{M}\int y\,dm \qquad z_{CM} = \frac{1}{M}\int z\,dm$$

CM of solid *symmetrical* objects

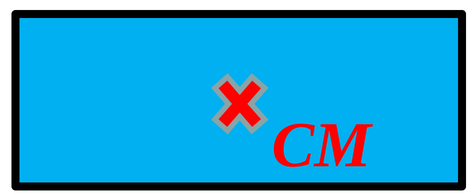
The easiest trick is to use symmetry



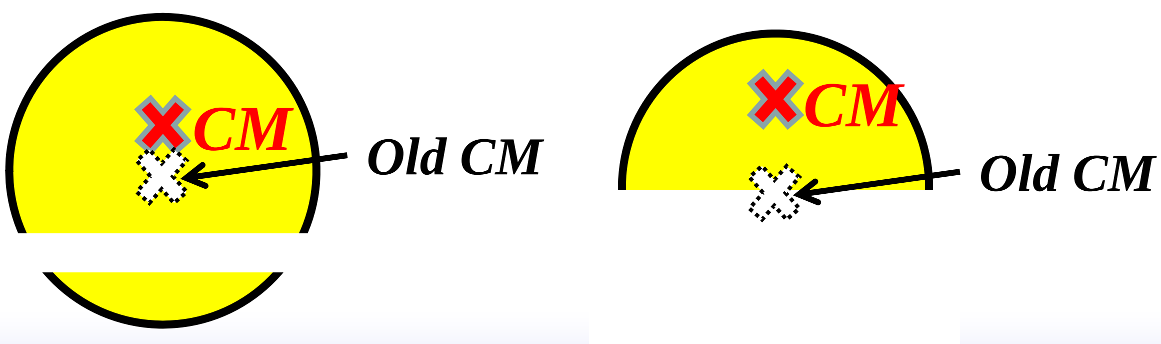
$$x_{CM} = \frac{\ldots + m(5) + m(-5) + \ldots}{M}$$

$$y_{CM} = \frac{\ldots + m(-4) + m(4) + \ldots}{M}$$

So, contributions to CM from symmetrical points cancel each other and, as a result, the CM coordinates are (0;0)



If we break a symmetry, the CM will be shifted



ConcepTest   Center of Mass

The disk shown below in (1) clearly has its center of mass at the center.

Suppose a smaller disk is cut out as shown in (2).

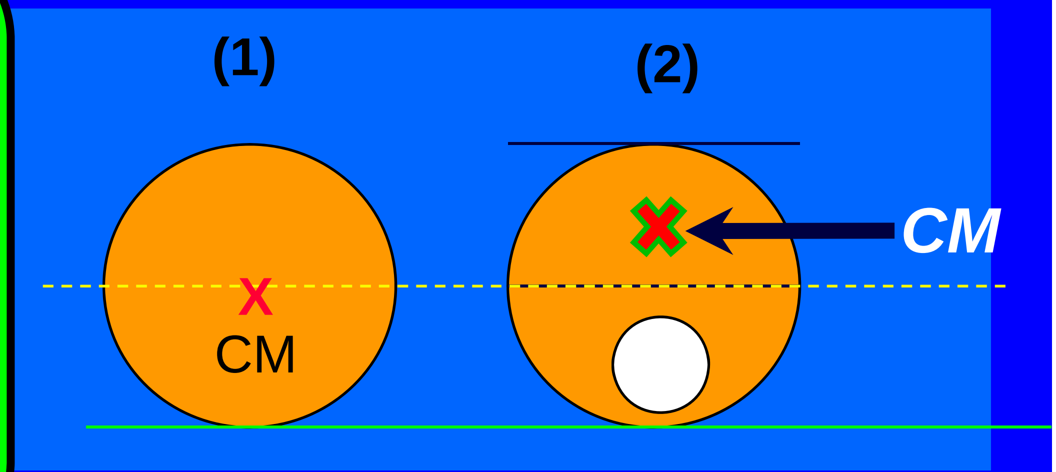
Where is the center of mass of (2) as compared to (1) ?

A) higher

B) lower

C) at the same place

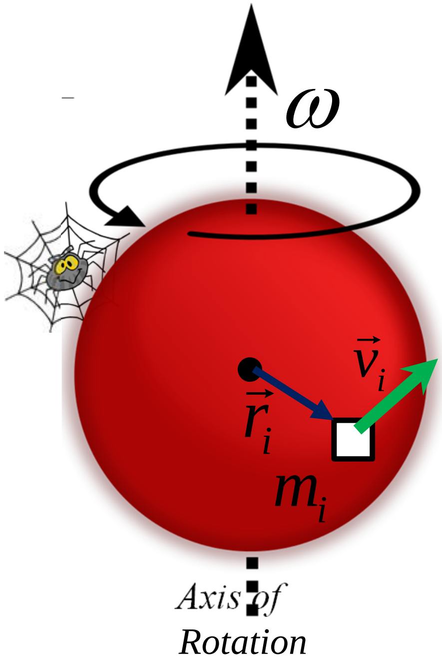
D) there is no definable CM in this case



Rotational Kinetic Energy and Moment of Inertia

Rotational Kinetic Energy

Consider a pure rotation of a solid object.



A rotating object has kinetic energy because all particles in the object are in motion.

Now, let's divide extended mass into smaller sections m_i

$$K_{rot} = \sum \tfrac{1}{2} m_i v_i^2 \qquad v_i = \omega r_i$$

$$K_{rot} = \sum \tfrac{1}{2} m_i (\omega r_i)^2 = \sum \tfrac{1}{2} m_i \omega^2 r_i^2$$

$$K_{rot} = \tfrac{1}{2} \omega^2 \sum m_i r_i^2$$

We ended up with a new structure. Let's give it a nice name and a symbol:

Moment of inertia $I = \sum m_i r_i^2$

Rotational Kinetic Energy

(The kinetic energy due to rotation is called rotational kinetic energy.)

$$K_{rot} = \frac{1}{2} I \omega^2$$

The units of moment of inertia are kg m^2.

Translational Kinetic Energy

$$K_{trans} = \frac{1}{2} m v^2$$

Moment of inertia is the rotational equivalent of mass.

$$m \leftrightarrow I$$
$$v \leftrightarrow \omega$$

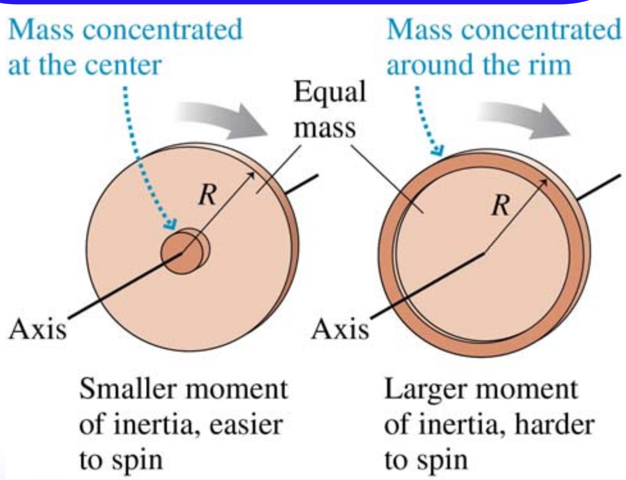
This is not a new form of energy, merely the familiar kinetic energy of motion written in a new way.

Why moment of inertia is needed to describe rotation?

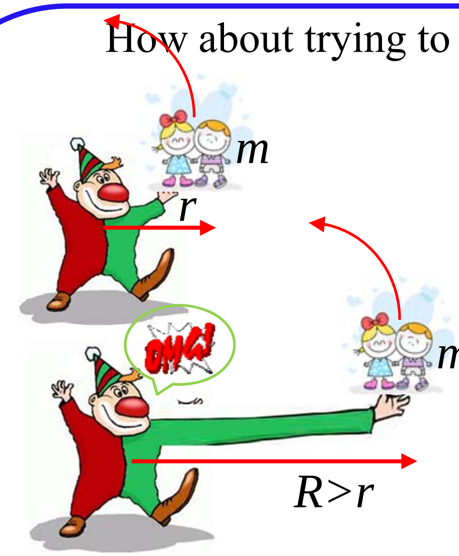
Let's try to move objects translationally?

difficult

We know that it is easier
to move a lighter object than heavier.
So information about mass is enough to describe the "translational challenge".



How about trying to rotate these kids up?



difficult

We know that it is easier
to rotate the same object when it is closer to an axis of rotation. So information about mass is NOT enough to describe the "rotational challenge".

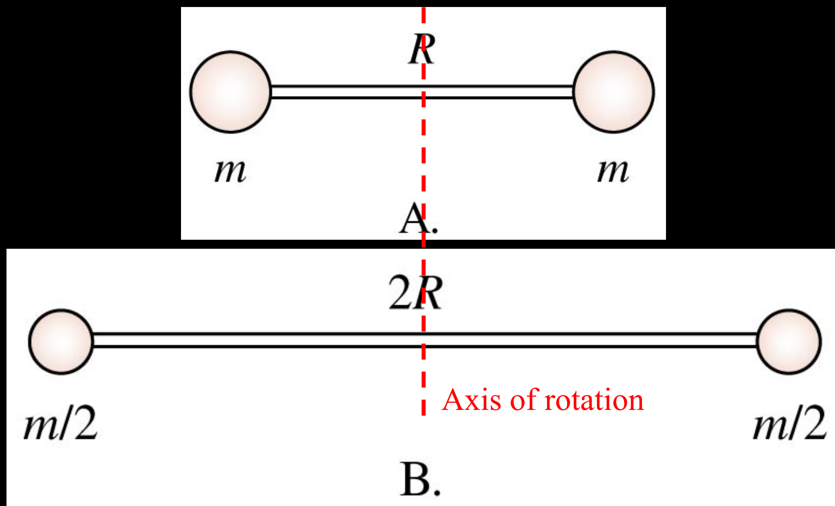
- So it should depend on *r and m*.

That is why the moment of inertia appeared in the "rotational game" , $I=mr^2$

Mass farther from the rotation axis contributes more to the moment of inertia than mass nearer the axis.
Moment of inertia is the rotational equivalent of mass.

ConcepTest Center of Mass

- Which dumbbell has the larger moment of inertia about the midpoint of the rod?
  The connecting rod is massless.



A) Dumbbell A.

B) Dumbbell B.

C) Their moments of inertia are the same.

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

Distance from the axis is more important than mass.

$$I_A = m(R/2)^2 + m(R/2)^2 = 2mR^2/4 = mR^2/2$$

$$I_B = (m/2)(R)^2 + (m/2)(R)^2 = 2mR^2/2 = mR^2$$

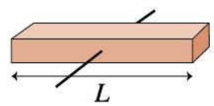
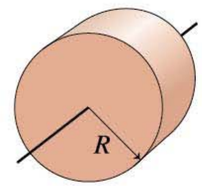
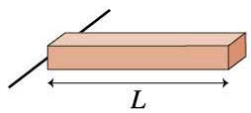
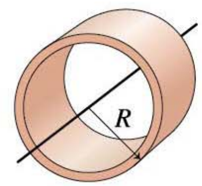
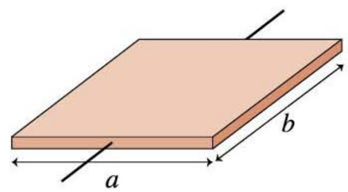
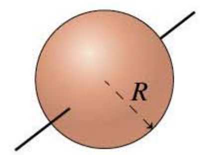
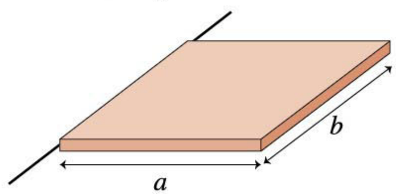
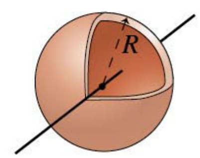
Moments of Inertia for many solid objects

As we did for center of mass, divide a solid object into many small cells of mass Δm and let $\Delta m \to 0$. The moment of inertia sum becomes

$$I = \sum_i r_i^2 \, \Delta m \xrightarrow[\Delta m \to 0]{} I = \int r^2 \, dm$$ where r is the distance from the rotation axis.

The procedure is much like calculating the center of mass.
One rarely needs to do this integral because moments of inertia of common shapes are tabulated.

TABLE 12.2 Moments of inertia of objects with uniform density

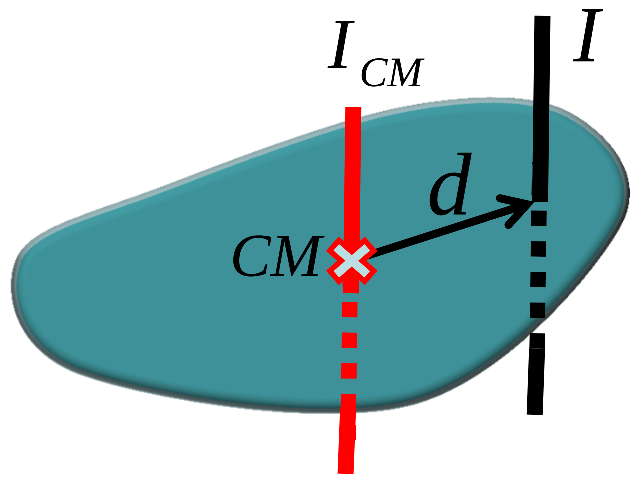
Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Parallel Axis Theorem (w/out proof)

Moment of inertia depends on the axis of rotation. $I = \sum m_i r_i^2$
How can we relate the moment of inertia measured relative to different axis of rotation?

The moment of inertia about any axis parallel to that axis through the center of mass is given by

$$I = I_{CM} + Md^2$$



I : moment of inertia about any parallel axis
I_{CM} : moment of inertia about an axis through its center of mass
M : total mass
d : distance from a parallel axis to the center of mass.

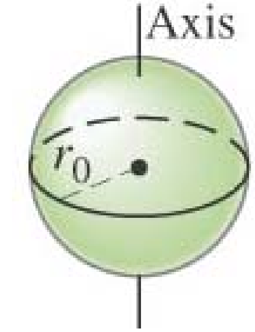
Relative to which axis is the moment of inertia the smallest?

BTW: The moment of inertia of any object about an axis through its center of mass is the minimum moment of inertia for an axis in that direction in space.

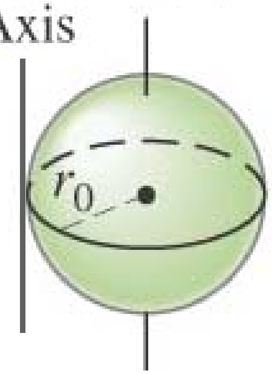
Example *Parallel Axis Theorem: Sphere*

For a uniform sphere of radius r_0

Moment of inertia for the sphere, rotating about an axis through its center of mass $I_{CM} = \frac{2}{5} M r_0^2$



Moment of inertia for the sphere about an axis going through the edge of the sphere?

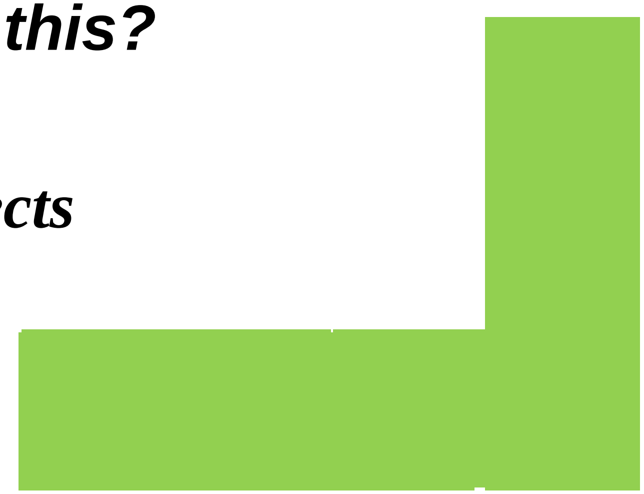


Apply Parallel Axis Theorem:

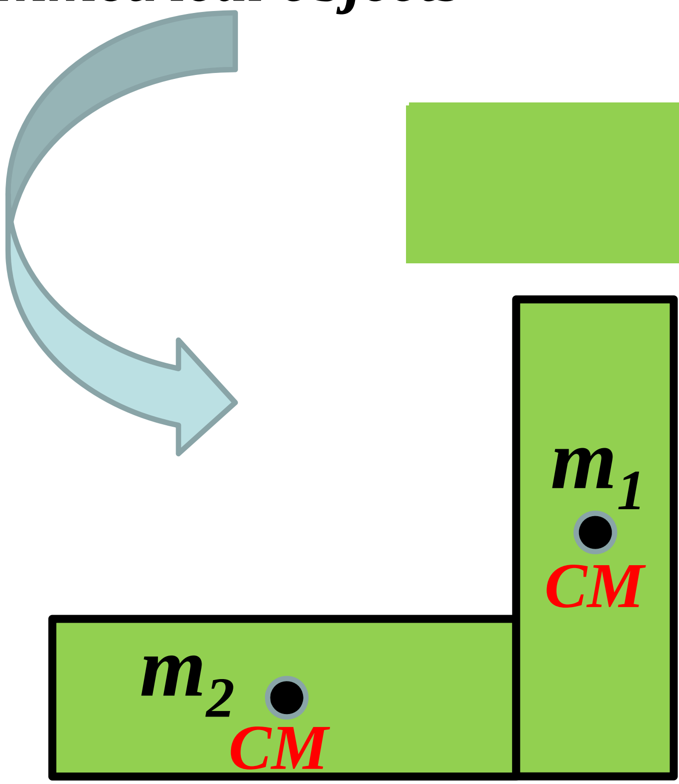
$$I = I_{CM} + Md^2 = \frac{2}{5} M r_0^2 + M r_0^2 = \frac{7}{5} M r_0^2$$

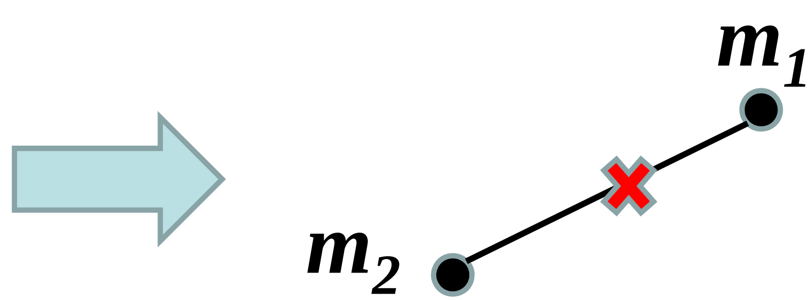
CM of Solid Objects *(nice trick)*

How to deal with objects like this?



Divide it into symmetrical objects





CM of the original object

Thank you
See you on Wednesday