Chapter 11:
First Order Circuits

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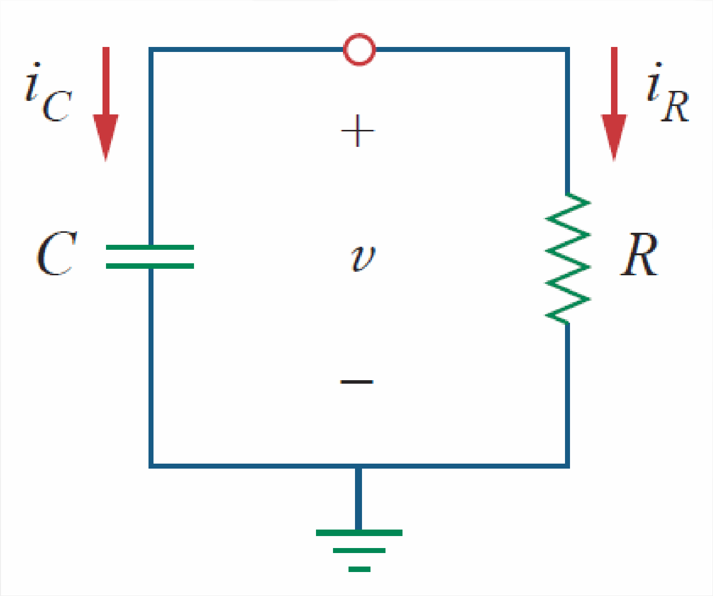
Introduction

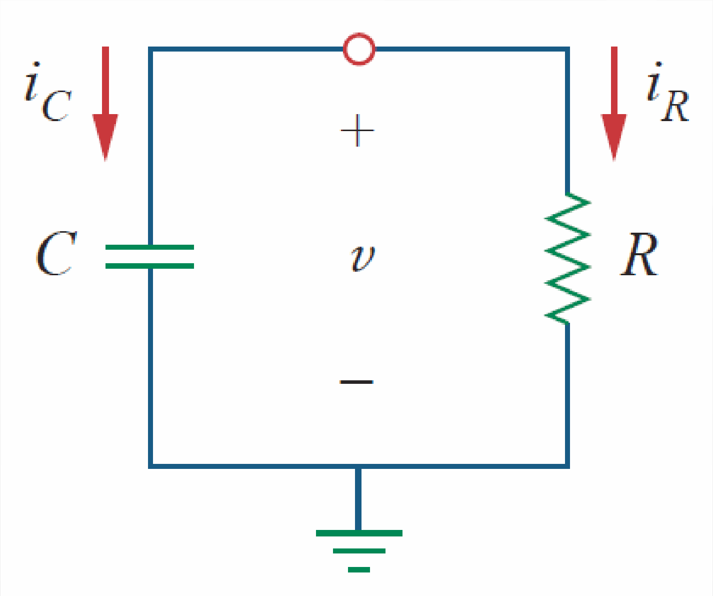
- A **first-order** circuit is characterized by a first-order differential equation

- Example:



The Source-Free RC Circuit

- A **source-free *RC*** circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.



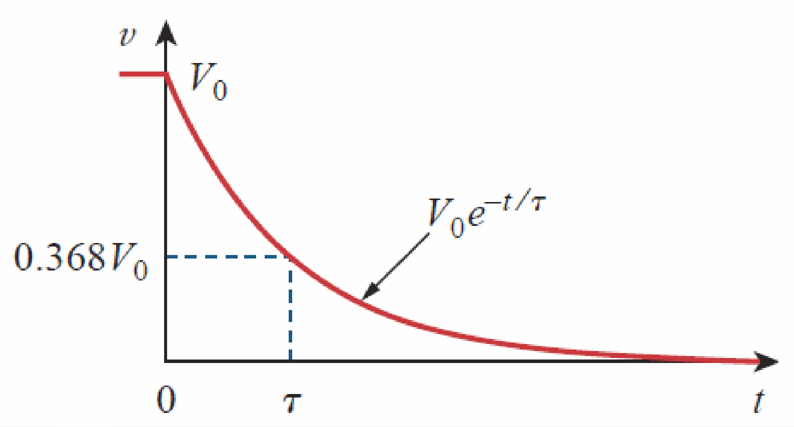
The Source-Free RC Circuit

- The capacitor is initially charged. At t=0, $v(0) = V_0$
- Energy stored is $w(0) = \frac{1}{2}CV_0^2$
- Applying KCL yields $i_C + i_R = 0$
- By definition: $i_C = C\frac{dv}{dt}$ and $i_R = \frac{v}{R}$ thus $C\frac{dv}{dt} + \frac{v}{R} = 0$
- Or $\frac{dv}{dt} + \frac{v}{RC} = 0$
- This is a ***first-order differential equation***
- We rearrange as : $\frac{dv}{v} = -\frac{1}{RC}dt$
- Integrating both sides: $\ln v = -\frac{\mathrm{t}}{\mathrm{RC}} + \ln \mathrm{A}$, A is a constant
- Thus $\ln\frac{\mathrm{v}}{\mathrm{A}} = -\frac{\mathrm{t}}{\mathrm{RC}}$
- Taking powers of e produces: $\mathrm{v(t)} = Ae^{-\frac{t}{RC}}$
- From the initial conditions: $\mathrm{v(t)} = V_0e^{-\frac{t}{RC}}$

The Source-Free RC Circuit

- The **natural response** of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

- The **time constant** of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.



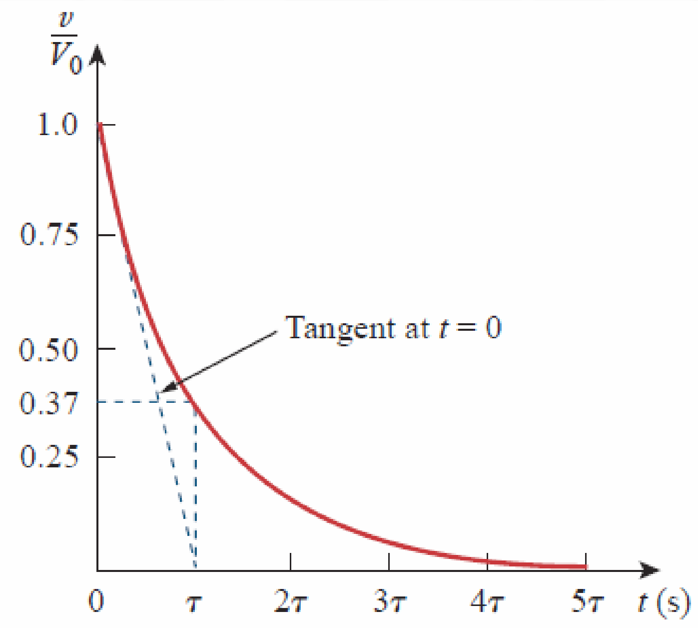
- At $t = \tau$, $V_0 e^{-\frac{t}{RC}} = V_0 e^{-1} = 0.368 V_0$

The Source-Free RC Circuit

$RC = \tau$ is the time constant

$$v(t) = V_0 e^{-\frac{t}{\tau}}$$

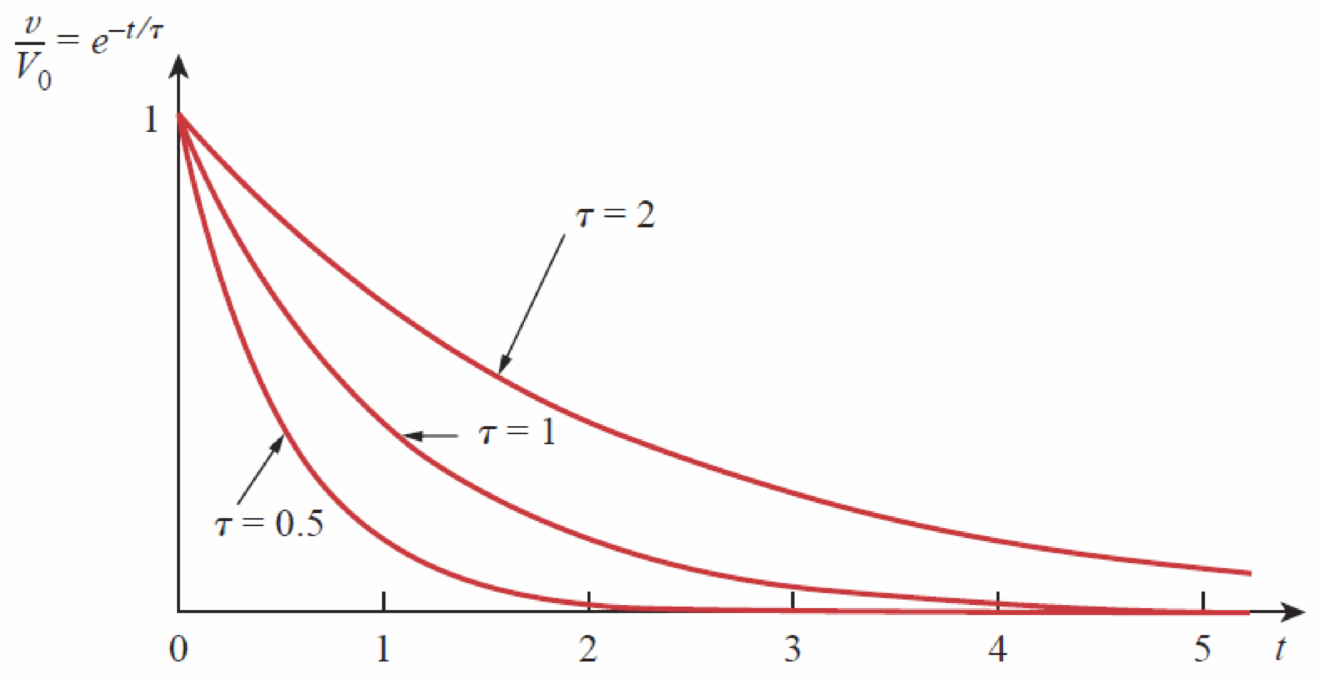
t	v(t)/V_0
T	0.36788
2τ	0.13534
3τ	0.04979
4τ	0.01832
5τ	0.00674



The Source-Free RC Circuit

- With the voltage we can find the current

$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}\, e^{-\frac{t}{\tau}}$$

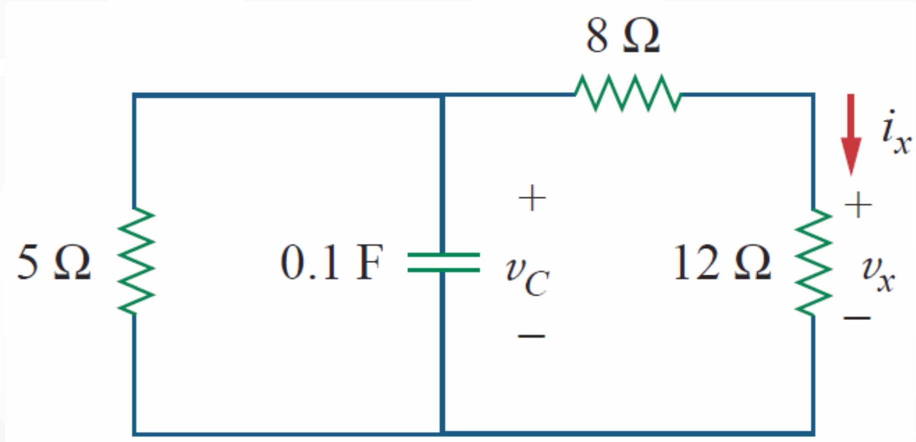


The Source-Free RC Circuit

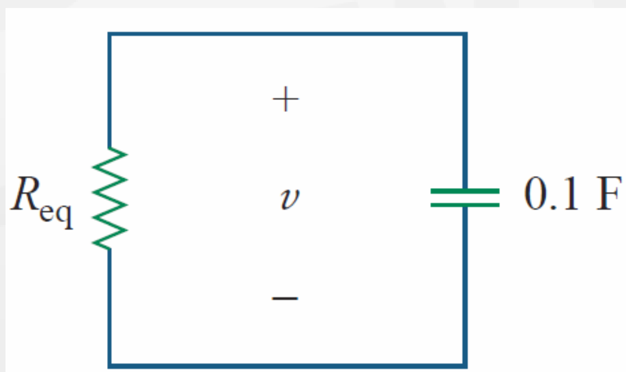
- The Key to Working with a Source-free RC Circuit is Finding:
- 1. The initial voltage $v(0) = V_0$ across the capacitor
- 2. The time constant τ

The Source-Free RC Circuit

- Example:
- Let $v_C(0) = 15V$, Find v_C, v_X and i_X for $t > 0$



- Solution:
- First find the equivalent resistance or the Thevenin resistance at the capacitor terminals. Our objective is always to first obtain capacitor voltage v_C



The Source-Free RC Circuit

- Solution:
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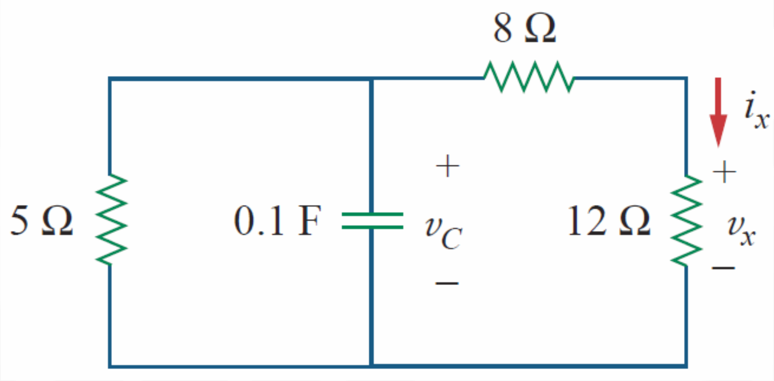
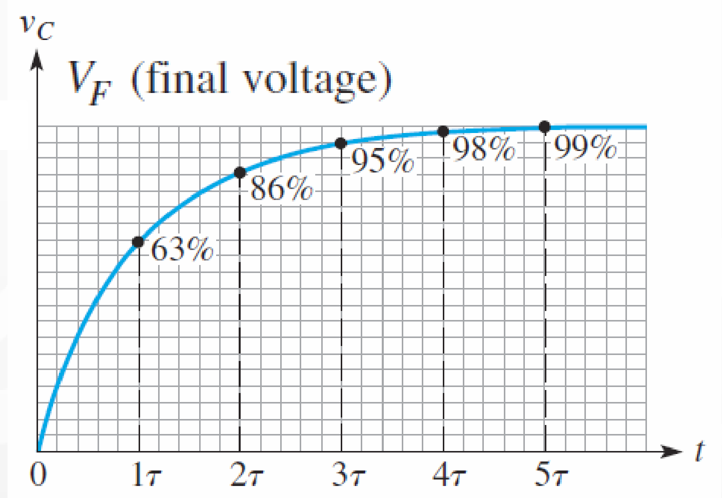


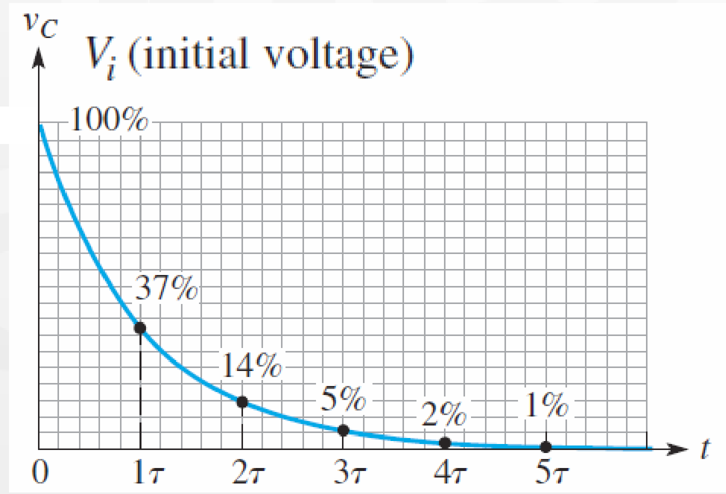
- $R_{eq} = \frac{20\times5}{20+5} = 4\ \Omega$
- $\tau = R_{eq}C = 4 \times 0.1 = 0.4$ s
- $\text{v} = v(0)e^{-\frac{t}{\tau}} = 15e^{-t/0.4}$ V
- $v_C = v = 15e^{-2.5t}$ V
- We use now the voltage divider rule to get v_X
- $v_X = \frac{12}{12+8}v = 0.6(15e^{-2.5t}) = 9e^{-2.5t}$ V

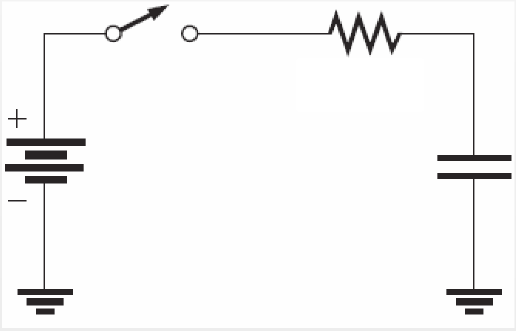
The RC Time Constant

- When a capacitor is charged Through a series resistor and dc source, the charging curve is exponential.

- When a capacitor is discharged through a resistor, the discharge curve is also an exponential.







Universal exponential curves

- The universal curves can be applied to general formulas for the voltage (or current) curves for RC circuits. The general voltage formula is

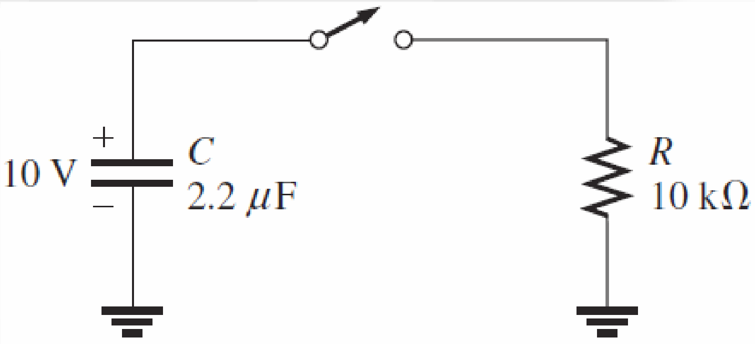
$$v_C = V_F + (V_i - V_F)e^{-\frac{t}{RC}}$$

- V_F = final value of voltage
- V_i = initial value of voltage
- v_C = instantaneous value of voltage

- The final capacitor voltage is greater than the initial voltage when the capacitor is charging, or less that the initial voltage when it is discharging.

Exercise

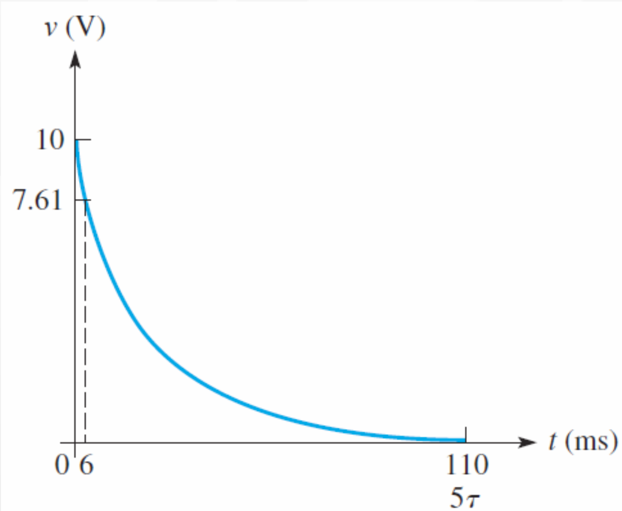
Determine the capacitor voltage at a point in time 6 ms after the switch is closed. Draw the discharging curve.



The time constant is

$$\tau = RC = (10k\Omega) \times (2.2\mu F) = 22ms$$

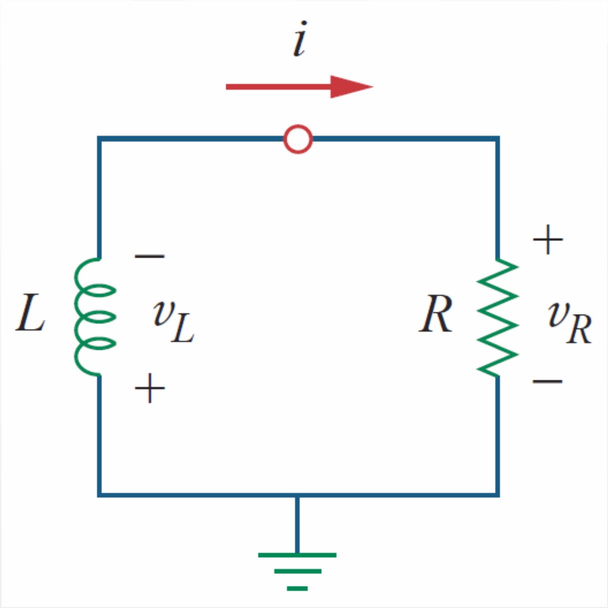
The initial capacitor voltage is 10 V.
Notice that 6 ms is less than one time constant, so the capacitor will discharge less than 63%. Therefore, it will have a voltage greater than 37% of the initial voltage at 6 ms.



$$v_C = V_i e^{-\frac{t}{RC}} = 10 \times e^{-\frac{6ms}{22ms}} = 7.61V$$

The Source-Free RL Circuit

- Consider the series connection of a resistor and an inductor. Our goal is to determine the circuit response, which we
- will assume to be the current *i(t)* through the inductor.



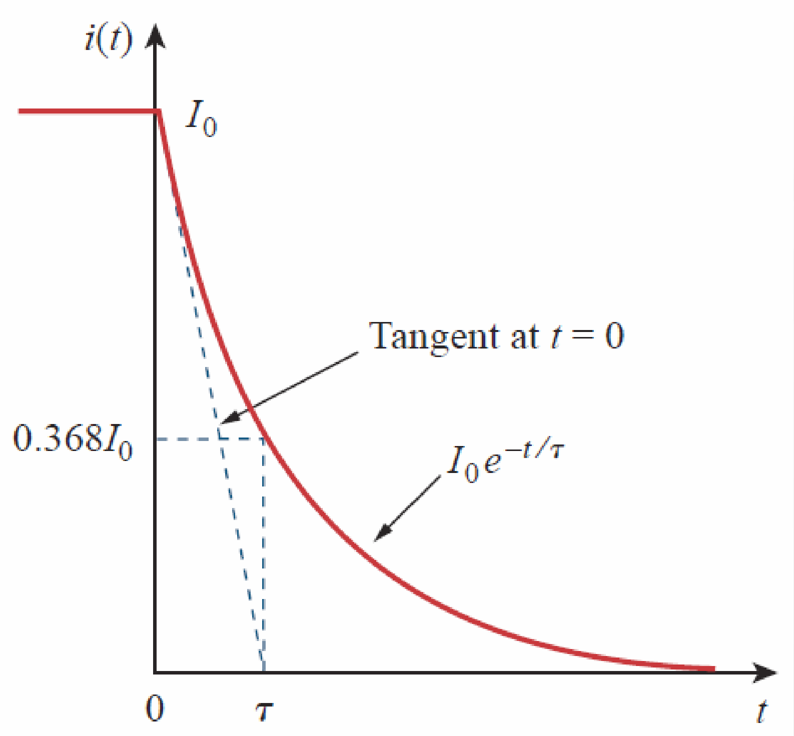
The Source-Free RL Circuit

- The inductor has an initial current at t=0, $i(0) = I_0$
- Energy stored is $w(0) = \frac{1}{2} L I_0^2$
- Applying KVL yields $v_L + v_R = 0$
- By definition: $v_l = L\frac{di}{dt}$ and $v_R = iR$ thus $\mathrm{L}\frac{di}{dt} + Ri = 0$
- Or $\frac{di}{dt} + \frac{R}{L} i = 0$
- We rearrange as : $\int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$
- Integrating gives: $\ln \mathrm{i}|_0^{i(t)} = -\frac{\mathrm{Rt}}{\mathrm{L}}|_0^t$, $\ln \mathrm{i(t)} - \ln I_0 = -\frac{Rt}{L} + 0$
- Thus $\ln \frac{\mathrm{i(t)}}{I_0} = -\frac{Rt}{\mathrm{L}}$
- Taking powers of e: $\mathrm{i(t)} = I_0 e^{-\frac{Rt}{L}}$

The Source-Free RL Circuit

$R/L = \tau$ is the time constant

$$i(t) = I_0 e^{-\frac{t}{\tau}}$$



The Source-Free RC Circuit

- With the current we can find the voltage across the resistor

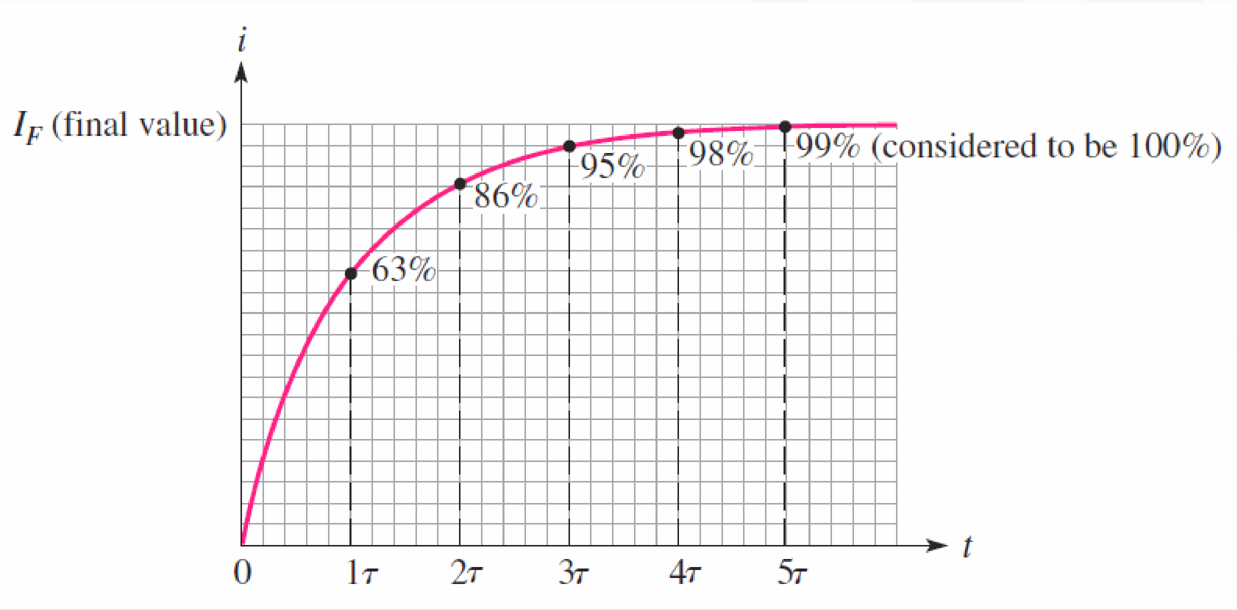
$$v_R(t) = i(t)R = I_0 R e^{-\frac{t}{\tau}}$$

- The Key to Working with a Source-free RL Circuit is Finding:
- 1. The initial current $i(0) = I_0$ through the inductor
- 2. The time constant τ

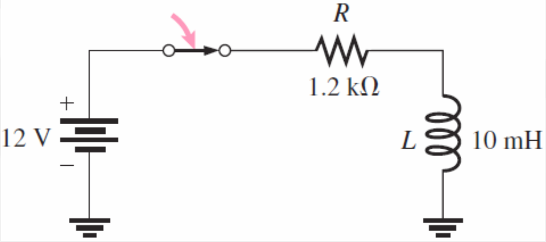
Inductor in DC Circuits

Increasing current

- In a series *RL* circuit, the current will increase to approximately 63% of its full value in one time-constant interval after voltage is applied.



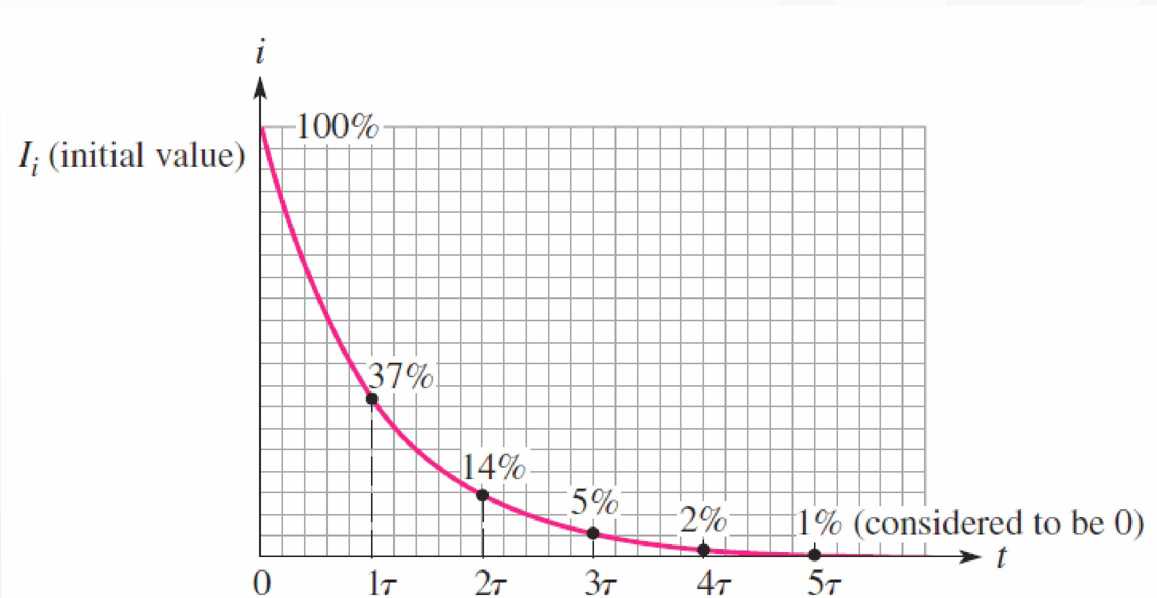
$$\tau = \frac{L}{R}$$

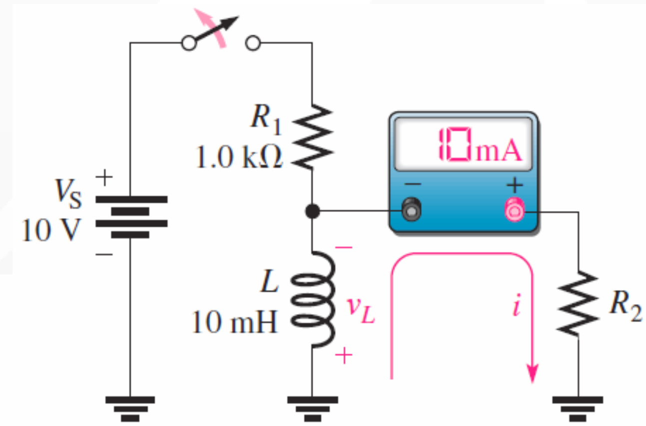


Inductor in DC Circuits

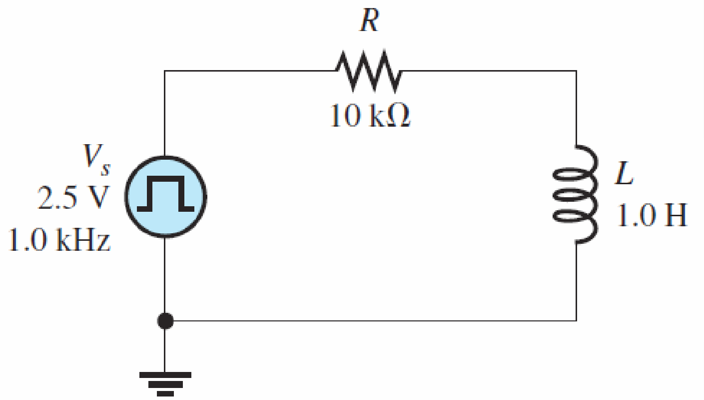
Decreasing current

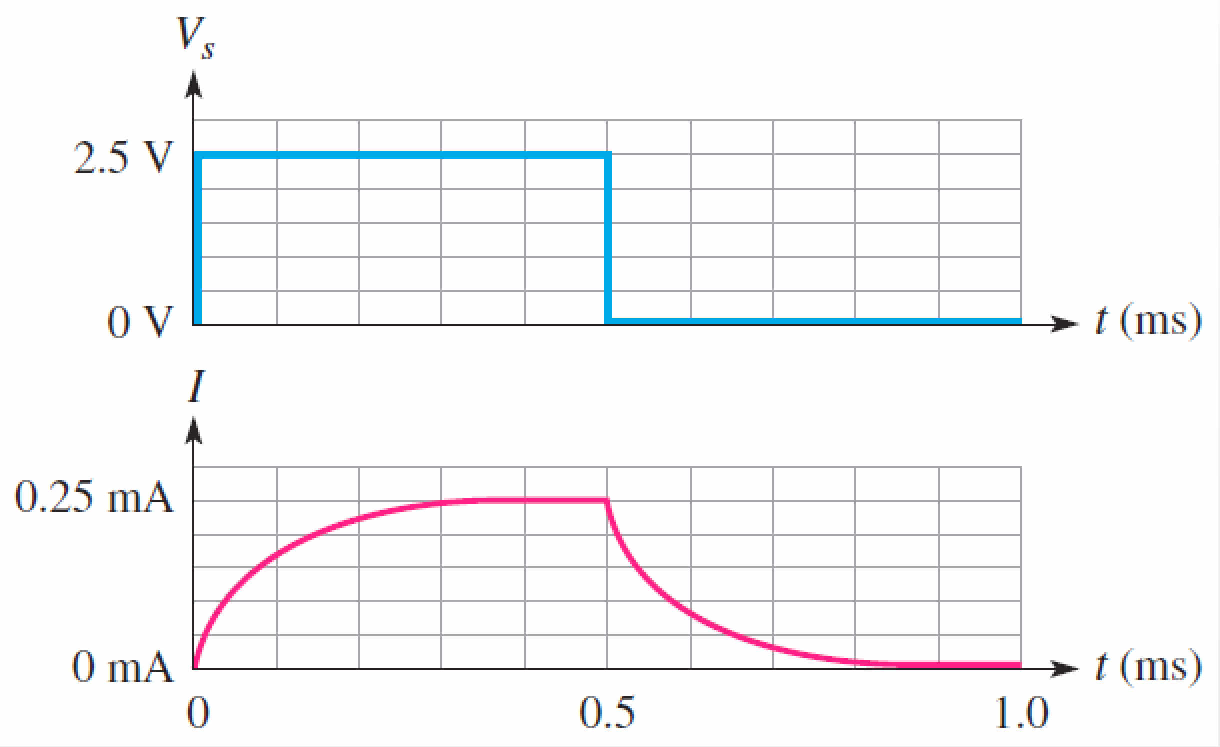
$$\tau = \frac{L}{R}$$





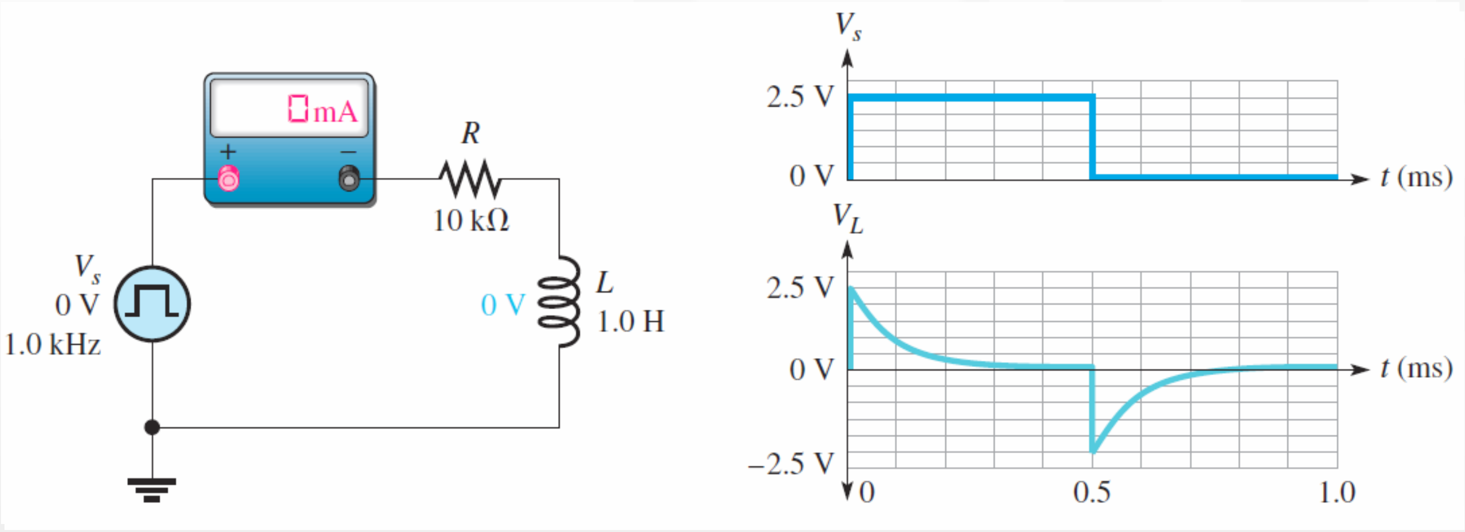
Response to a Square Wave





Voltages in the Series *RL* Circuits

- Generator: square wave voltage
- V_L: Voltage across the inductance



The exponential Formulas

$$v = V_F + (V_i - V_F)e^{-\frac{Rt}{L}}$$

$$i = I_F + (I_i - I_F)e^{-\frac{Rt}{L}}$$

- Increasing Current (I_i=0)

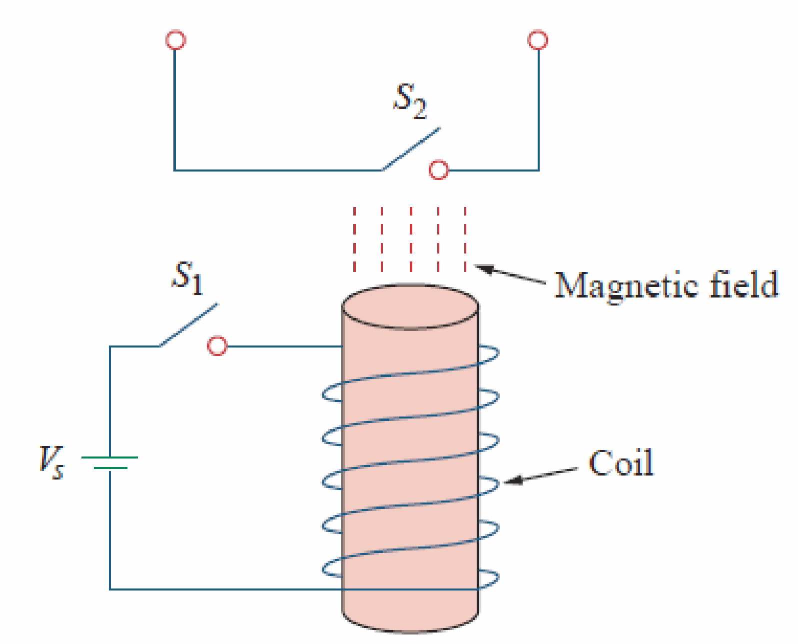
$$i = I_F(1 - e^{-\frac{Rt}{L}})$$

- Decreasing Current (I_F=0)

$$i = I_i e^{-\frac{Rt}{L}}$$

Problem

The coil of a certain relay is operated by a 12-V battery. If the coil has a resistance of 150 Ω and an inductance of 30 mH and the current needed to pull in is 50 mA, calculate the relay delay time.



Solution:

The current through the coil is given by

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

where

$$i(0) = 0, \qquad i(\infty) = \frac{12}{150} = 80\text{ mA}$$

$$\tau = \frac{L}{R} = \frac{30 \times 10^{-3}}{150} = 0.2\text{ ms}$$

Thus,

$$i(t) = 80[1 - e^{-t/\tau}]\text{ mA}$$

If $i(t_d) = 50$ mA, then

$$50 = 80[1 - e^{-t_d/\tau}] \quad \Rightarrow \quad \frac{5}{8} = 1 - e^{-t_d/\tau}$$

or

$$e^{-t_d/\tau} = \frac{3}{8} \quad \Rightarrow \quad e^{t_d/\tau} = \frac{8}{3}$$

By taking the natural logarithm of both sides, we get

$$t_d = \tau \ln\frac{8}{3} = 0.2 \ln\frac{8}{3}\text{ ms} = 0.1962\text{ ms}$$