Chapter 12: Sinusoids and Phasors

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Sinusoids

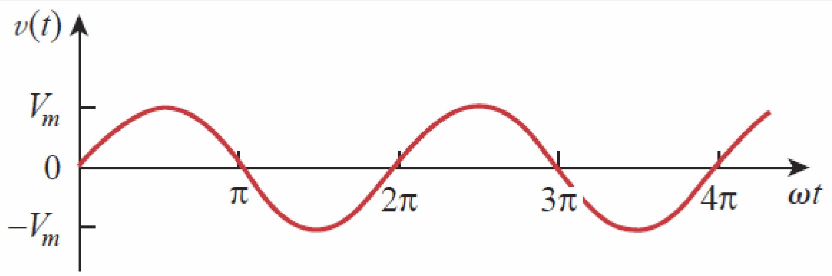
- Consider the sinusoidal voltage

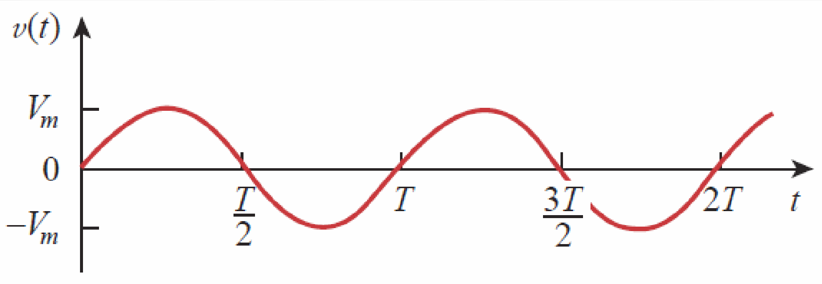
$$v(t) = V_m \sin \omega t$$

- Where
  - V_m = the amplitude of the sinusoid
  - ω = the angular frequency in radians/s
  - ωt = the argument of the sinusoid

- The period is

$$T = \frac{2\pi}{\omega}$$

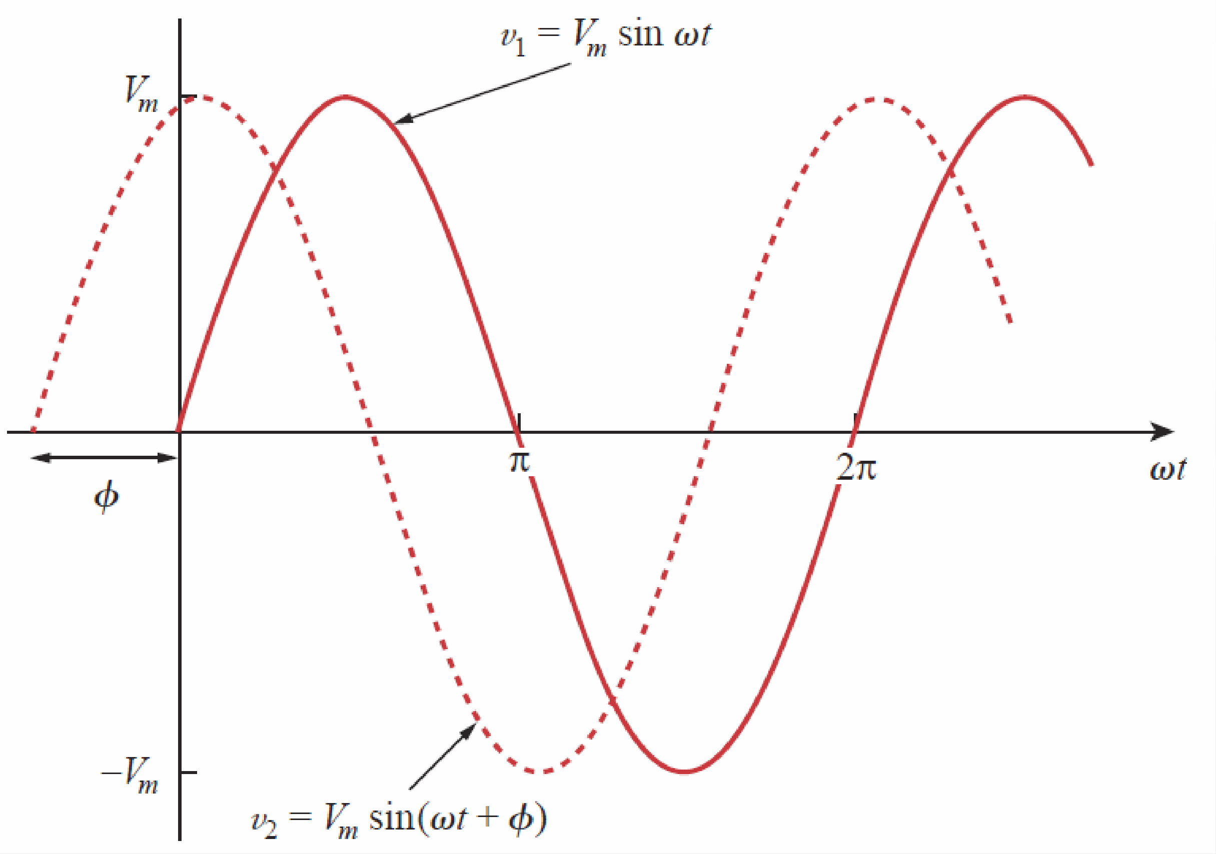




Sinusoids

- The more general expression is

$$v(t) = V_m \sin(\omega t + \phi)$$



Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid

Complex Numbers

- A complex number z can be written in rectangular form as:

$$z = x + jy$$

- Where $j = \sqrt{-1}$; x is the real part of z; y is the imaginary part of z

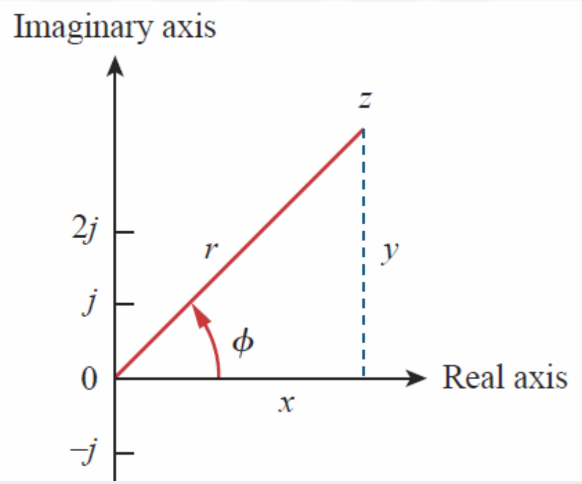
- z can also be written in exponential form as:

$$z = r\, e^{j\phi}$$

- The relationship between the rectangular and the exponential form is:

$$r = \sqrt{x^2 + y^2}, \quad \phi = tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$



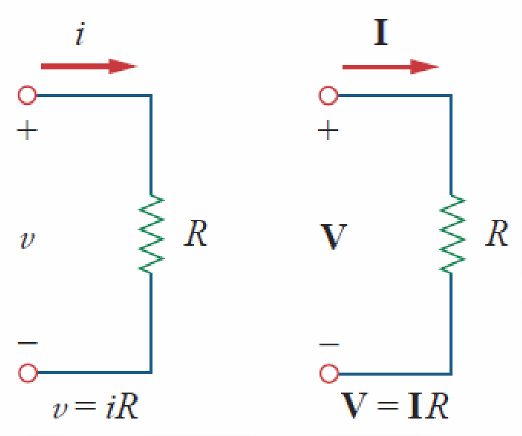
Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid

- Let’s see the phasor relationship for circuit elements

phasor relationship for circuit elements

Resistor

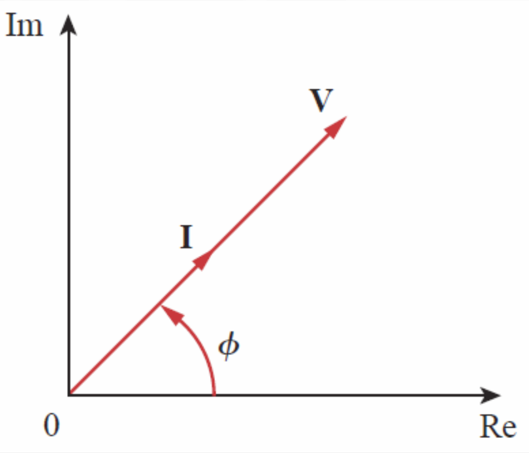


- If the current through a resistor R is $i = I_m \cos(\omega t + \phi)$, the voltage across it is given by Ohm's law as

$$v = iR = RI_m \cos(\omega t + \phi)$$

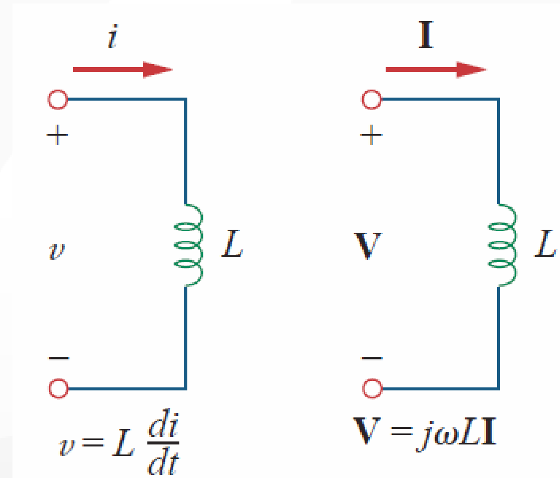
- The phasor representation in frequency domain is

$$\mathbf{V} = R\mathbf{I}$$



phasor relationship for circuit elements

Inductor



- If the current through an inductor L is $i = I_m \cos(\omega t + \phi)$, the voltage across it is

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \phi)$$

- With $-\sin A = \cos(A + 90^o)$, we got

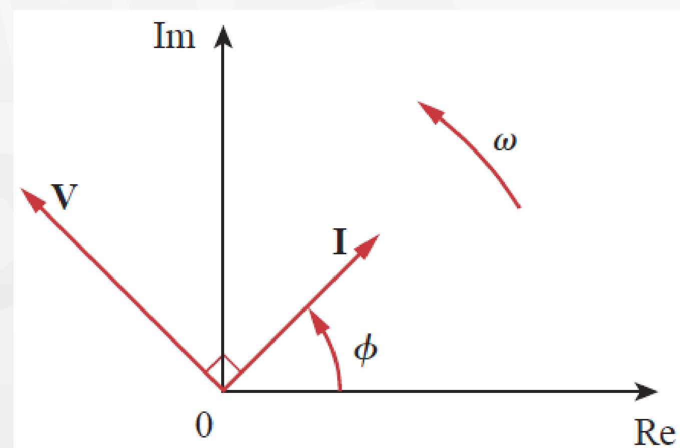
$$v = \omega L I_m \cos(\omega t + \phi + 90^\circ)$$

- Which transform to the phasor

$$\mathbf{V} = \omega L I_m e^{j(\phi + 90^\circ)}$$

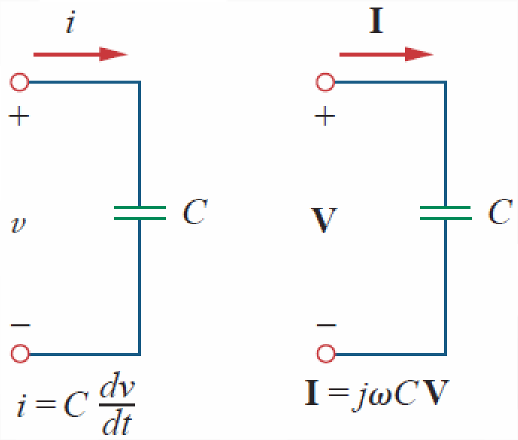
- Thus

$$\mathbf{V} = j\omega L\mathbf{I}$$



phasor relationship for circuit elements

Capacitor

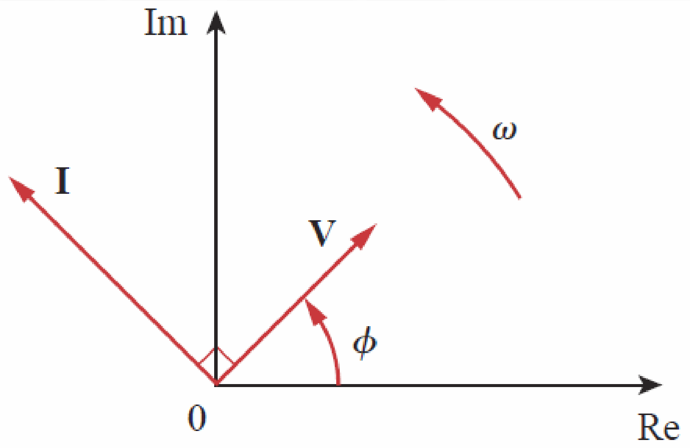


- For the Capacitor *C*, assume the voltage across it is $v = V_m \cos(\omega t + \phi)$. The current through the capacitor is

$$i = C\frac{dv}{dt}$$

- In frequency we obtain

$$\mathbf{I} = j\omega C\mathbf{V} \quad \Rightarrow \quad \mathbf{V} = \frac{\mathbf{I}}{j\omega C}$$



Impedance and Admittance

- From the table we can write the ratio of the phasor voltage and the phasor current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \qquad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}$$

- The impedance **Z** of a circuit is the ratio of the phasor voltage **V** to the phasor current **I**, measured in ohms (Ω)

$$\boldsymbol{Z} = \frac{\mathbf{V}}{\mathbf{I}}, \qquad \mathbf{V} = \mathbf{Z}\mathbf{I}$$

- As a complex quantity, the impedance may be expressed as

$$\mathbf{Z} = R + jX = |\boldsymbol{Z}|e^{\theta}$$

$$|\boldsymbol{Z}| = \sqrt{R^2 + X^2}, \qquad \theta = tan^{-1}\frac{X}{R}$$

$$R = |\boldsymbol{Z}|\cos\theta, \qquad X = |\boldsymbol{Z}|\sin\theta$$

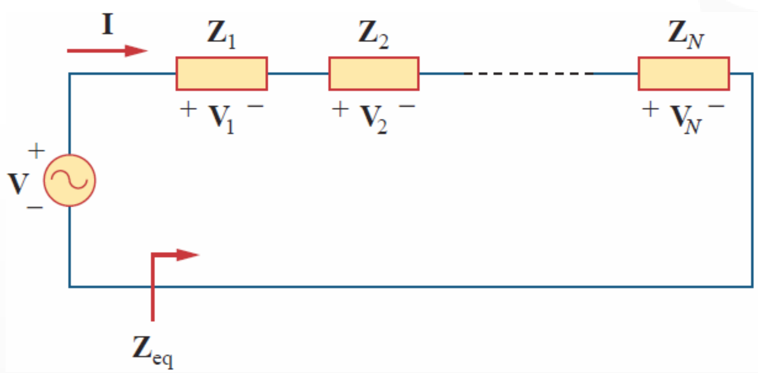
Summary

Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = V\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

Element	Impedance	Admittance
R	$\mathbf{Z} = R$	$\mathbf{Y} = \dfrac{1}{R}$
L	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = \dfrac{1}{j\omega L}$
C	$\mathbf{Z} = \dfrac{1}{j\omega C}$	$\mathbf{Y} = j\omega C$

Impedance Combination

N impedance in series

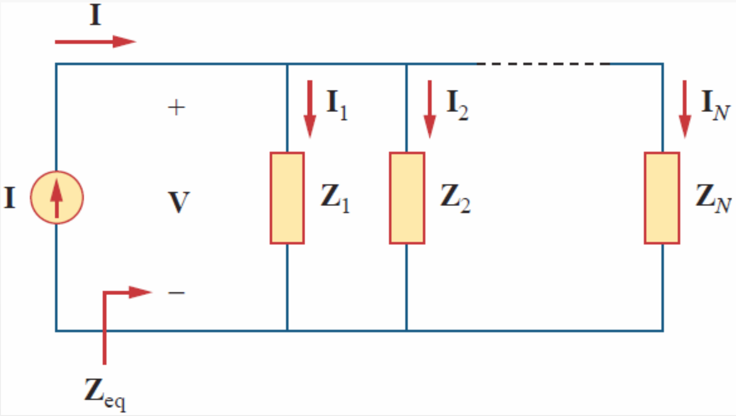


$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N$$

$$\boxed{\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N}$$

N impedance in parallel



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right)$$

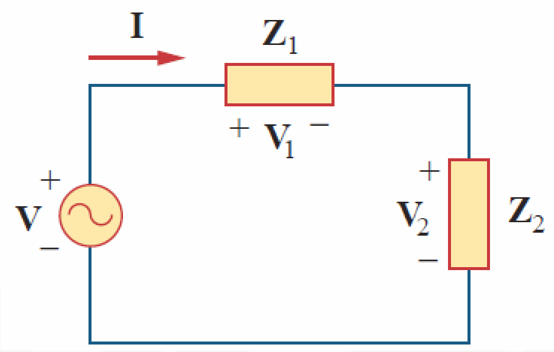
$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}$$

$$\boxed{\mathbf{Y}_{eq} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_N}$$

Impedance Combination

If N=2, the current through the series impedances is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

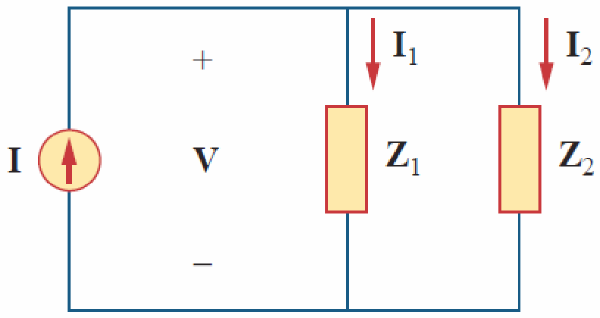


Since $\mathbf{V}_1=\mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2=\mathbf{Z}_2\mathbf{I}$

$$\mathbf{V}_1 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}, \qquad \mathbf{V}_2 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

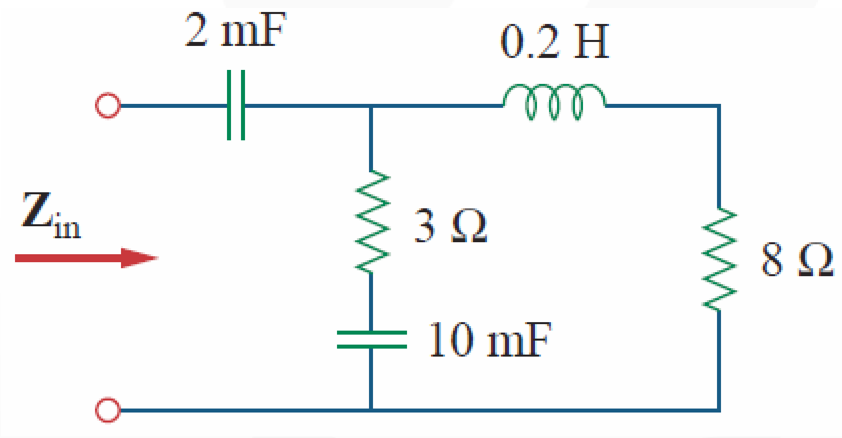
If N=2, the current through the parallel impedances is

$$\mathbf{Z}_{eq} = \frac{1}{\mathbf{Y}_{eq}} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2} = \frac{1}{1/\mathbf{Z}_1 + 1/\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$



$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}, \qquad \mathbf{I}_2 = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

Exercice



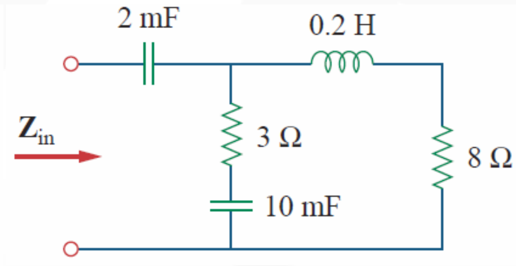
- Find the input impedance of the circuit. Assume that the circuit operates at $\omega = 50$ rad/s

Solution

Let $\mathbf{Z}_1$ = Impedance of the 2-mF capacitor

$\mathbf{Z}_2$ = Impedance of the 3-Ω resistor in series with the10-mF capacitor

$\mathbf{Z}_3$ = Impedance of the 0.2-H inductor in series with the 8-Ω resistor



Then

$$\mathbf{Z}_1 = \frac{1}{j\omega C} = \frac{1}{j50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$\mathbf{Z}_2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

$$\mathbf{Z}_3 = 8 + j\omega L = 8 + j50 \times 0.2 = (8 + j10)\ \Omega$$

The input impedance is

$$\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3 = -j10 + \frac{(3 - j2)(8 + j10)}{11 + j8}$$

$$= -j10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j10 + 3.22 - j1.07\ \Omega$$

Thus

$$\mathbf{Z}_{in} = 3.22 - j11.07\ \Omega$$