Medical Diagnostic Imaging

ECE16511/411

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Signals and Systems

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Introduction

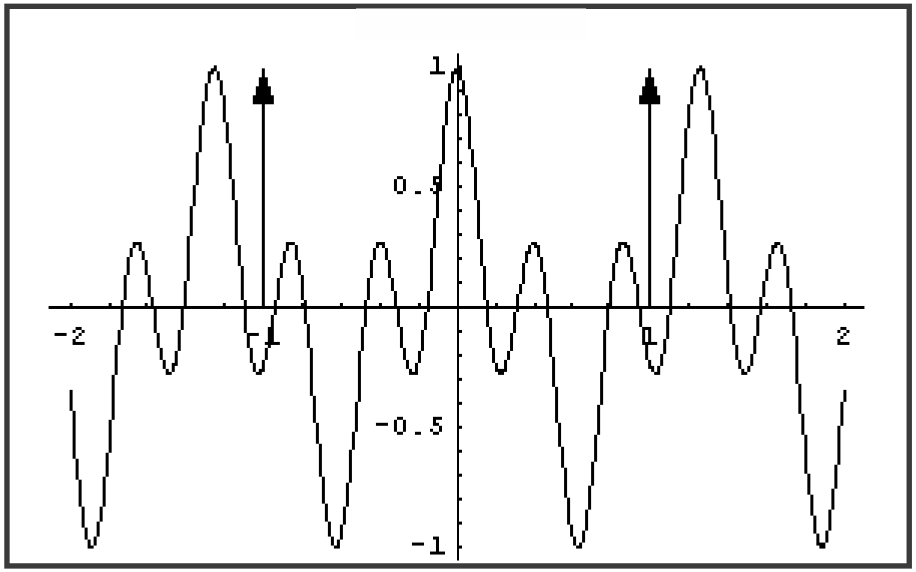
Signals
mathematical functions of one or more independent variables, capable of modeling a variety of physical processes.

Systems
respond to signals by producing new signals.

Signals can be classified into three categories:

1) Continuous Signal:

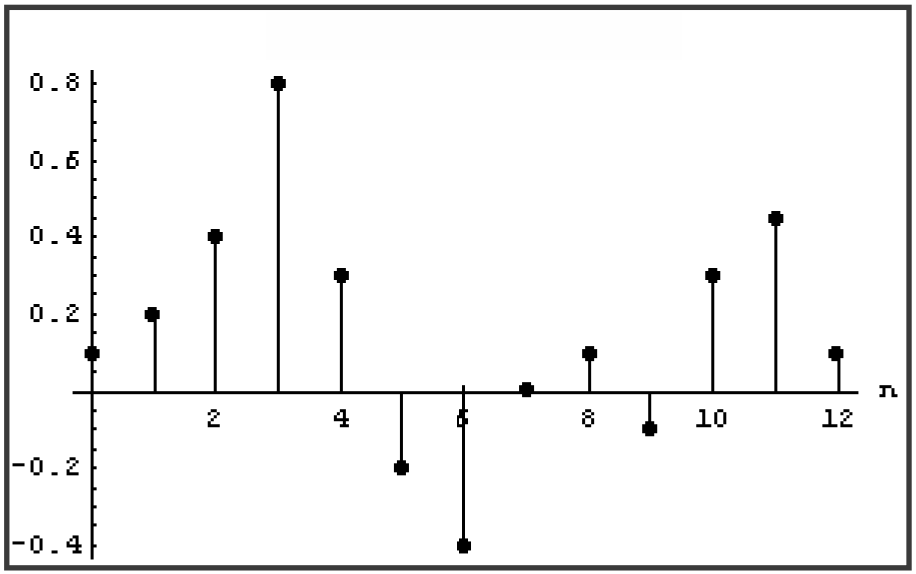
is a function of independent variables that range over a continuum of values.



Signals can be classified into three categories:

2) Discrete signal:

is a function of independent variables that range over discrete values



3) Mixed signal:

is a function of some continuous and some discrete independent variables

Systems can also be classified into three categories.

1) **Continuous-to-continuous system:**

   responds to a continuous signal by producing a continuous signal

2) **Continuous-to-discrete system:**

   responds to a continuous signal by producing a discrete signal.

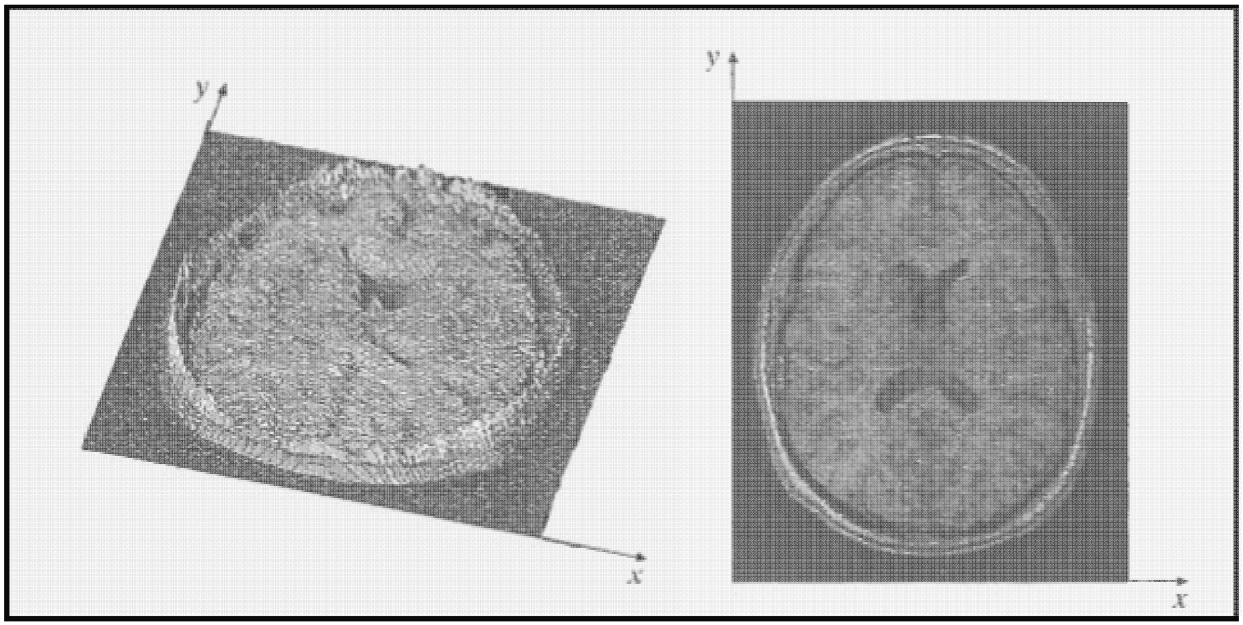
3) **Discrete-to-discrete system:**
   responds to a discrete signal by producing a discrete signal.

Signals

2-D continuous signal is defined as

$$f(x, y), \qquad -\infty \leq x,\ y \leq \infty$$

This signal can be represented (visualized) in two different ways.

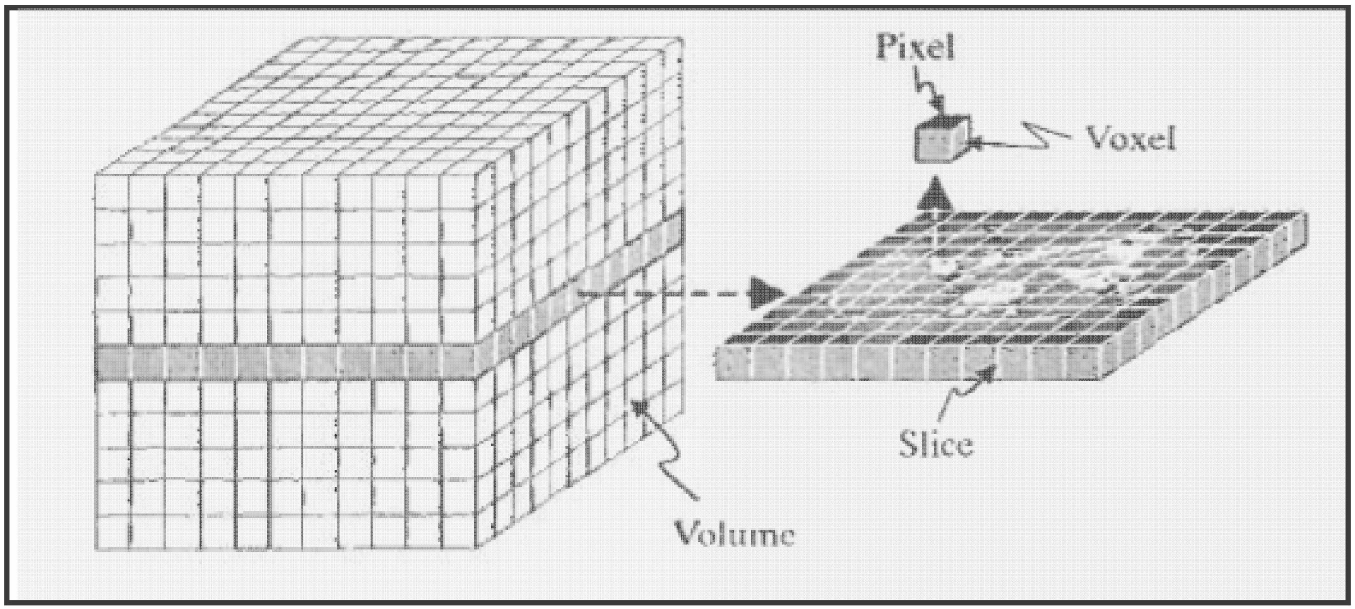


Functional Plot

Image display

Signals

2D and 3D display



Point Impulse

Point Impulse is used to model the concept of a point source, which is used in the characterization of imaging system resolution.

The point impulse is also known as the *delta function,* the *Dirac function and* the *impulse function*.

1D point impulse is defined as:

$$\delta(x) = 0, \quad x \neq 0,$$
$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0).$$

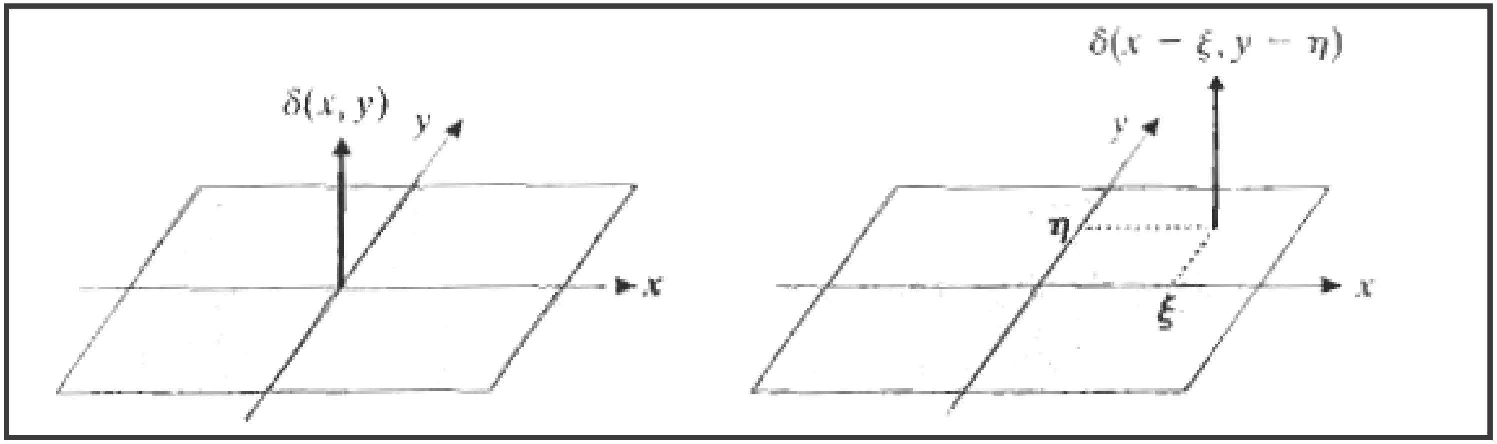
2D point impulse is defined as :

$$\delta(x, y) = 0, \quad (x, y) \neq (0, 0),$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x, y)\, dx\, dy = f(0, 0).$$

Point Impulse

Shifted 2-D point impulse is defined as:

$$f(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x - \xi, y - \eta)\, dx\, dy\,.$$



Point Impulse

Properties:

1) The product of a function with a point impulse as is another point impulse whose volume is equal to the value of the function at the location of the point impulse.

$$g(x,y) = f(x,y)\delta(x-\xi, y-\eta) = f(\xi,\eta)\delta(x-\xi, y-\eta)$$

2) Scaling

$$\delta(ax, by) = \frac{1}{|ab|}\delta(x,y)$$

3) Even Function

$$\delta(-x,-y) = \delta(x,y)$$

Line Impulse

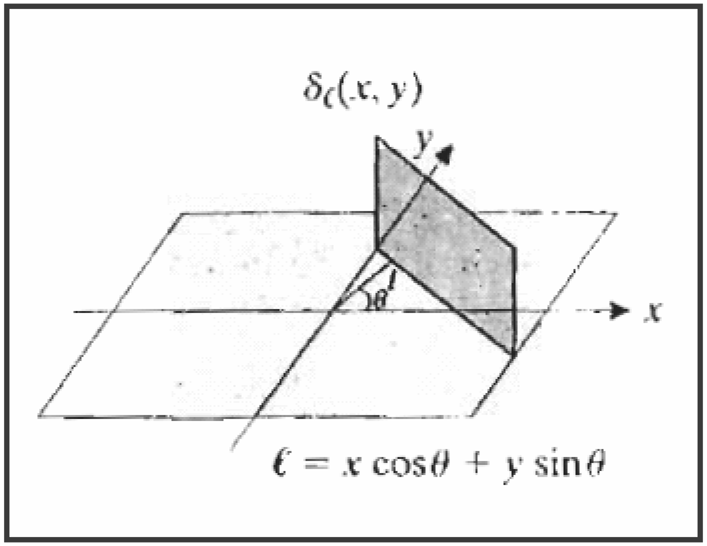
A set of points given by

$$L(\ell, \theta) = \{(x, y) \mid x\cos\theta + y\sin\theta = \ell\}$$

Defines a line whose unit normal is oriented at an angle θ relative to the x-axis and is at distance *l* from the origin in the direction of the unit normal.

The line impulse δ*l*(x, y) associated with line L(i, θ) is given by

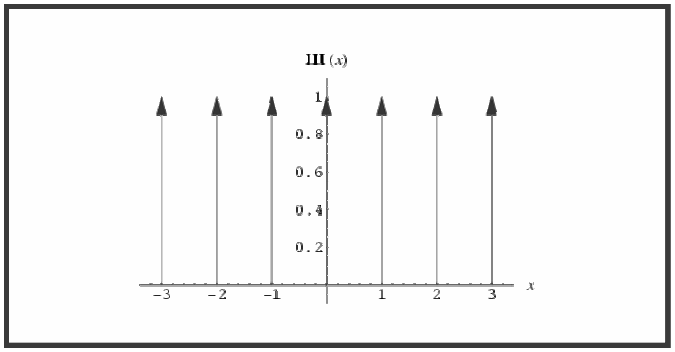
$$\delta_\ell(x, y) = \delta(x\cos\theta + y\sin\theta - \ell)$$



Comb (Shah) Function

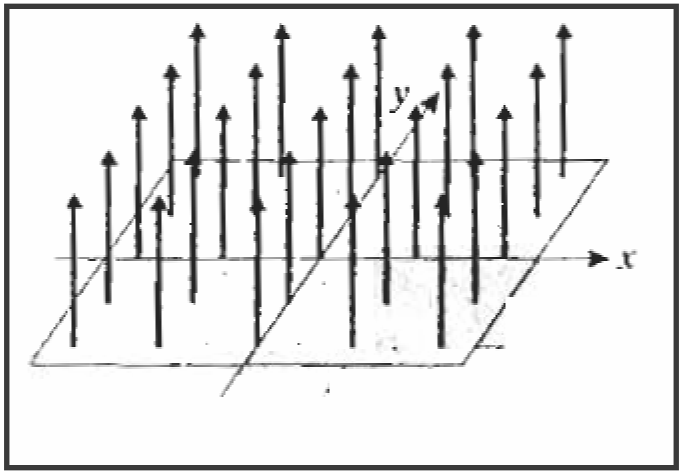
1D Comb function is

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



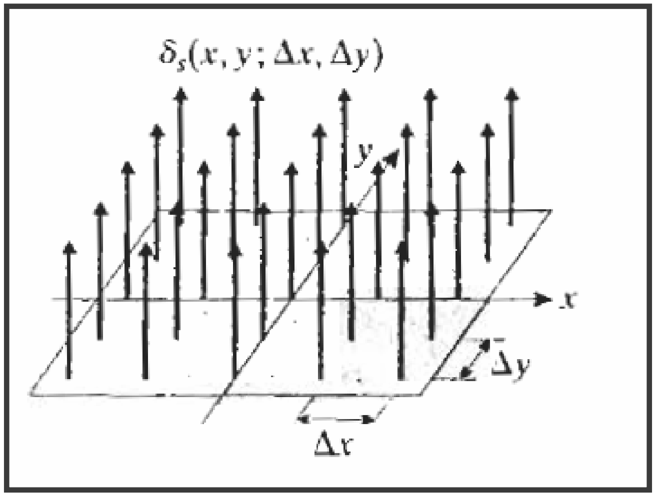
2D Comb function is

$$\mathrm{comb}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$



The sampling function

$$\delta_s(x, y; \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



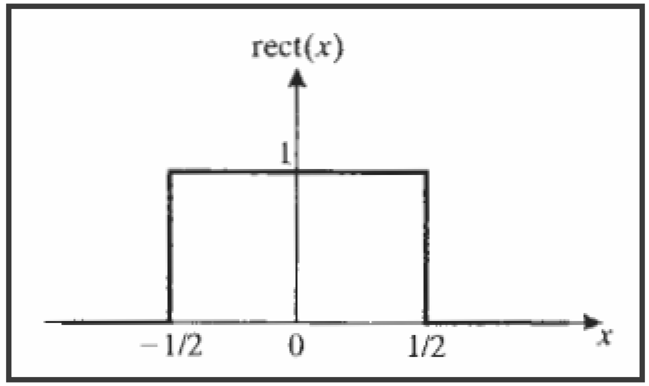
The sampling function related to the comb function is

$$\delta_s(x, y; \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \mathrm{comb}\left(\frac{x}{\Delta x}, \frac{y}{\Delta y}\right)$$

Rect Functions

The 1D rect function is given by

$$\mathrm{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0, & \text{for } |x| > \frac{1}{2} \end{cases} .$$



The 2D rect function is given by

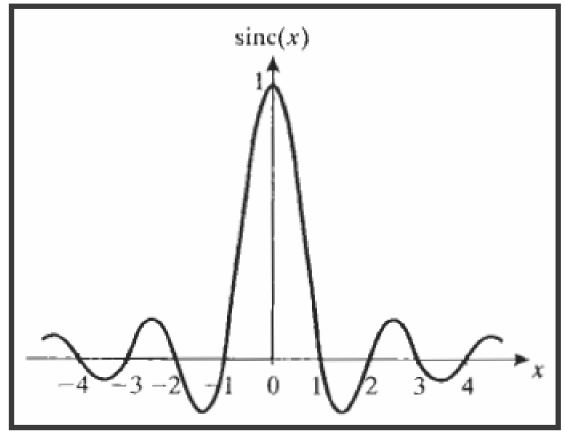
$$\mathrm{rect}(x, y) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \text{ and } |y| < \frac{1}{2} \\ 0, & \text{for } |x| > \frac{1}{2} \text{ or } |y| > \frac{1}{2} \end{cases}$$

$$\mathrm{rect}(x, y) = \mathrm{rect}(x)\,\mathrm{rect}(y)$$

Sinc Functions

The 1D sinc function is given by

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



The 2D *sinc function* is given by

$$\mathrm{sinc}(x, y) = \begin{cases} 1, & \text{for } x = y = 0 \\ \dfrac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}, & \text{otherwise.} \end{cases}$$

$$\mathrm{sinc}(x, y) = \mathrm{sinc}(x)\,\mathrm{sinc}(y)$$

Exponential and Sinusoidal Signals

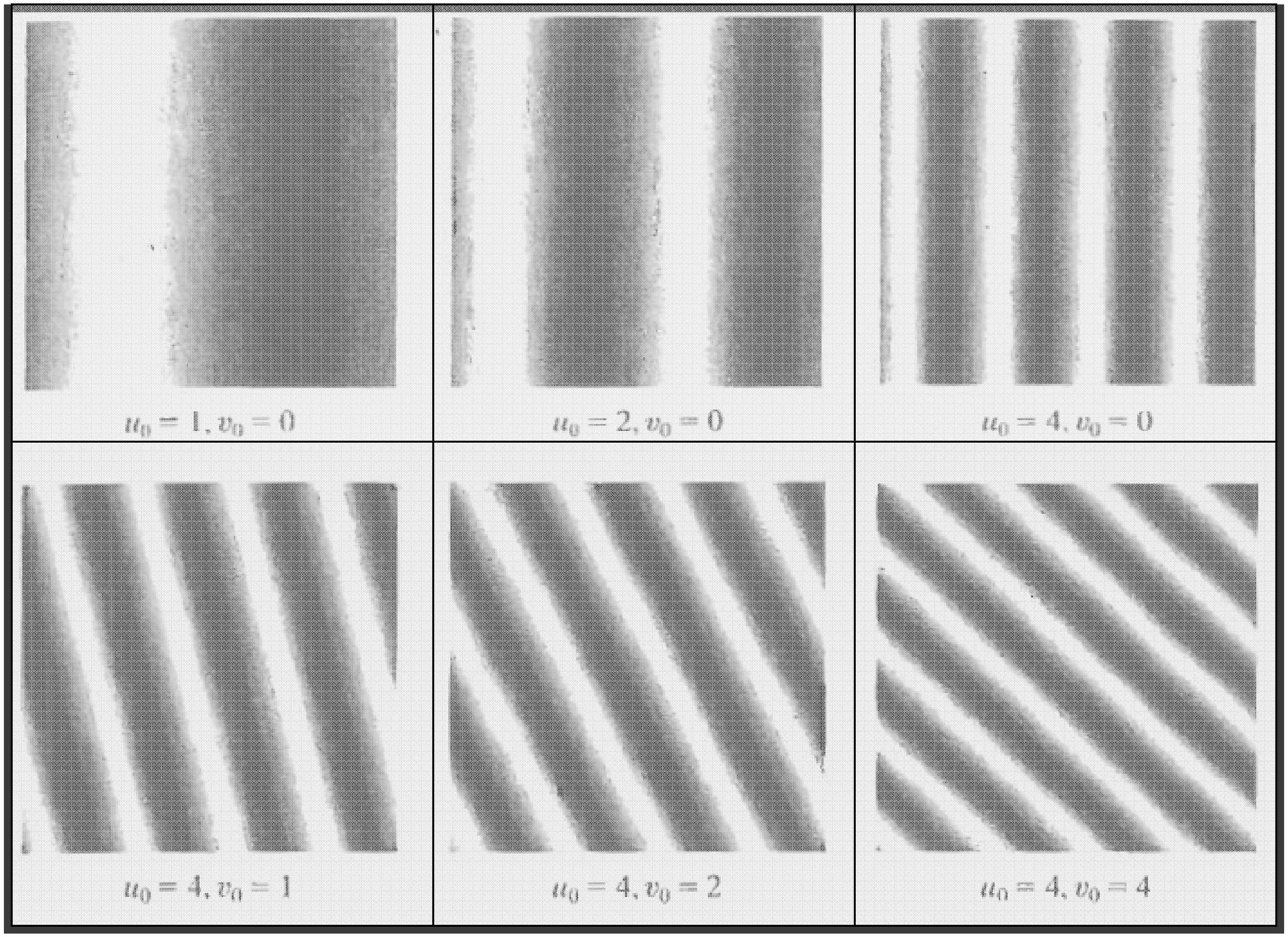
The complex exponential signal is

$$
\begin{aligned}
e(x, y) &= e^{j2\pi(u_0 x + v_0 y)} \\
&= \cos[2\pi(u_0 x + v_0 y)] + j \sin[2\pi(u_0 x + v_0 y)] \\
&= c(x, y) + js(x, y) .
\end{aligned}
$$

where

$$s(x, y) = \sin[2\pi(u_0 x + v_0 y)] \quad \text{and} \quad c(x, y) = \cos[2\pi(u_0 x + v_0 y)]$$

Exponential and Sinusoidal Signals



$$s(x, y) = \sin[2\pi(u_o x + v_o y)]$$

Separable Signals

A 2D signal f*(x, y)* is *separable* if there exist two 1D signals $f_1(x)$ and $f_2(y)$ such that

$$f(x, y) = f_1(x) f_2(y)$$

$$\text{rect}(x, y) = \text{rect}(x)\,\text{rect}(y)$$

$$\text{sinc}(x, y) = \text{sinc}(x)\,\text{sinc}(y),$$

Operating on separable signals is much simpler than operating on purely 2D signals.

Periodic Signals

A signal f*(x, y)* is a *periodic signal* if there exist two positive constants X and Y such that

$$f(x,y) = f(x+X,y) = f(x,y+Y)$$

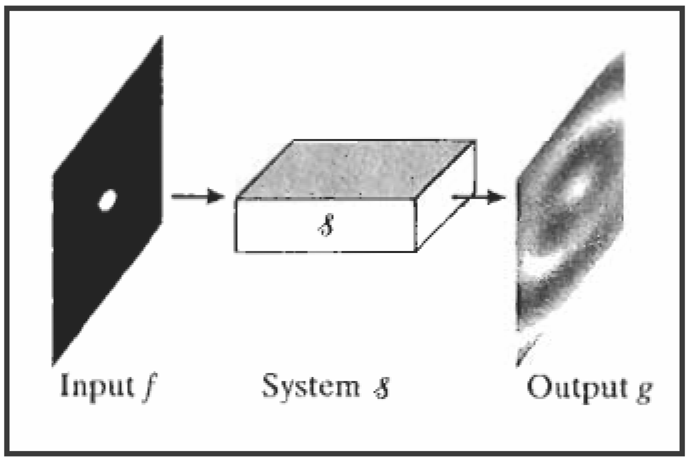
- The sampling function $\delta_s(x, y; \Delta x, \Delta y)$ is a periodic signal with periods $X = \Delta x$ and $Y = \Delta y$.

- The exponential and sinusoidal signals are periodic with periods $X = 1/u_o$ and $Y = 1/v_o$.

Systems

A *continuous-to-continuous* (or simply *continuous) system* is defined as a transformation £ of an *input* continuous signal *f(x,y)* to an *output* continuous signal *g(x, y).*

$$g(x, y) = \mathcal{S}[f(x, y)]$$



Linear Systems

A system £ is a *linear system* if, when the input consists of a weighted summation of several signals, the output will also be a weighted summation of the responses of the system to each individual input signal.

$$\mathcal{S}\left[\sum_{k=1}^{K} w_k f_k(x, y)\right] = \sum_{k=1}^{K} w_k \mathcal{S}\left[f_k(x, y)\right]$$

It is a system that satisfy the superposition principle.

Impulse Response

The point spread function (PSF), or the impulse response function of a system is the output of a system to a point impulse.

$$h(x, y; \xi, \eta) = \mathcal{S}[\delta_{\xi\eta}(x, y)]$$

For Linear systems

$$\begin{aligned}
g(x, y) &= \mathcal{S}[f(x, y)] \\
&= \mathcal{S}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\delta(x - \xi, y - \eta)\, d\xi\, d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{S}\left[f(\xi, \eta)\delta_{\xi\eta}(x, y)\right]\, d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\,\mathcal{S}[\delta_{\xi\eta}(x, y)]\, d\xi\, d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)h(x, y; \xi, \eta)\, d\xi\, d\eta
\end{aligned}$$

Shift Invariance

A system is shift-invariant if an arbitrary translation of the input results in an identical translation in the output.

For a SIS the application of a shifted input
Gives a shifted output

$$f_{x_0 y_0}(x, y) = f(x - x_0, y - y_0)$$

$$g(x - x_0, y - y_0) = \mathcal{S}[f_{x_0 y_0}(x, y)]$$

For a Linear Shift Invariant System (LSI)

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x - \xi, y - \eta)\, d\xi\, d\eta = h(x, y) * f(x, y)$$

Example1

Check if the system $g(x, y) = 2f(x, y)$ is LSI

Solution

I. Linearity

If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$g'(x, y) = 2\left(\sum_{k=1}^{K} w_k f_k(x, y)\right),$$

$$= \sum_{k=1}^{K} w_k 2 f_k(x, y),$$

$$= \sum_{k=1}^{K} w_k g_k(x, y),$$

II. Shift Invariance

$$g'(x, y) = 2f(x - x_0, y - y_0) = g(x - x_0, y - y_0)$$

Example2

Check if the system $g(x,y) = xyf(x,y)$ is LSI

Solution

I. Linearity

If $g'(x,y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x,y)$, then

$$g'(x,y) = xy\left(\sum_{k=1}^{K} w_k f_k(x,y)\right),$$

$$= \sum_{k=1}^{K} w_k xy f_k(x,y),$$

$$g'(x,y) = \sum_{k=1}^{K} w_k g_k(x,y),$$

II. Shift Invariance

$$g'(x,y) = xyf(x-x_0, y-y_0),$$

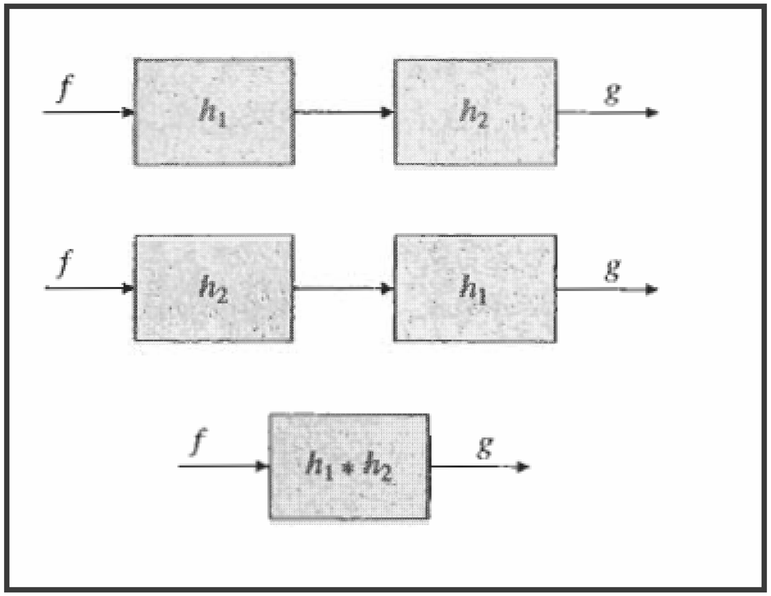
$$\neq (x-x_0)(y-y_0)f(x-x_0, y-y_0).$$

$$g'(x,y) \neq g(x-x_0, y-y_0),$$

Connections of LSI Systems

Cascade (Serial)

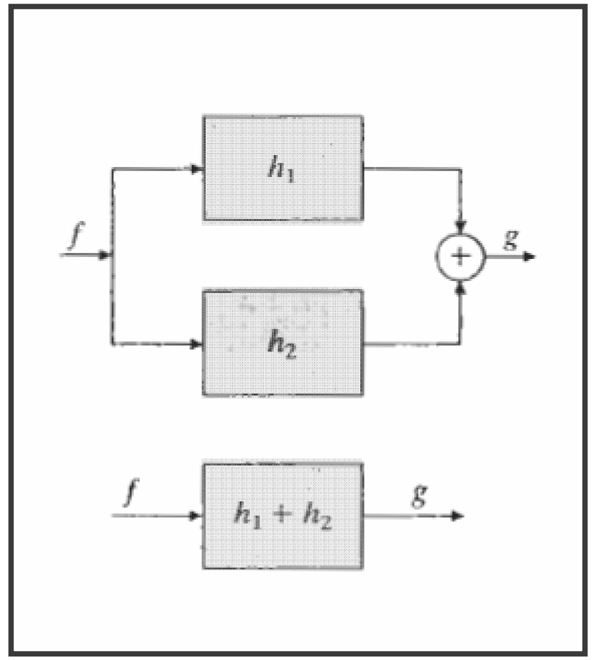
$$
\begin{aligned}
g(x,y) &= h_2(x,y) * [h_1(x,y) * f(x,y)], \\
&= h_1(x,y) * [h_2(x,y) * f(x,y)], \\
&= [h_1(x,y) * h_2(x,y)] * f(x,y),
\end{aligned}
$$



Connections of LSI Systems

Parallel

$$g(x,y) = h_1(x,y) * f(x,y) + h_2(x,y) * f(x,y)\,,$$
$$= [h_1(x,y) + h_2(x,y)] * f(x,y)\,,$$



Example3

Find the point spread function for the LSI system $h_1(x,y)*h_2(x,y)$ where

$$h_1(x,y) = \frac{1}{2\pi\sigma_1^2} e^{-(x^2+y^2)/2\sigma_1^2} \qquad \text{and} \qquad h_2(x,y) = \frac{1}{2\pi\sigma_2^2} e^{-(x^2+y^2)/2\sigma_2^2}$$

Solution

$$\begin{aligned}
h(x,y) &= h_1(x,y) * h_2(x,y)\,, \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\xi,\eta) h_1(x-\xi, y-\eta)\, d\xi\, d\eta\,, \\
&= \frac{1}{4\pi^2\sigma_1^2\sigma_2^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\xi^2+\eta^2)/2\sigma_2^2} e^{-[(x-\xi)^2+(y-\eta)^2]/2\sigma_1^2}\, d\xi\, d\eta\,, \\
&= \frac{1}{4\pi^2\sigma_1^2\sigma_2^2} \int_{-\infty}^{\infty} e^{-\xi^2/2\sigma_2^2-(x-\xi)^2/2\sigma_1^2}\, d\xi \\
&\qquad \int_{-\infty}^{\infty} e^{-\eta^2/2\sigma_2^2-(y-\eta)^2/2\sigma_1^2}\, d\eta\,.
\end{aligned}$$

Example3

Solution, cont.

$$\int_{-\infty}^{\infty} e^{-\xi^2/2\sigma_2^2-(x-\xi)^2/2\sigma_1^2}\,d\xi = e^{-x^2/2(\sigma_1^2+\sigma_2^2)}\int_{-\infty}^{\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left\{\xi-[\sigma_2^2/(\sigma_1^2+\sigma_2^2)]x\right\}^2}\,d\xi\,,$$

$$= e^{-x^2/2(\sigma_1^2+\sigma_2^2)}\int_{-\infty}^{\infty} e^{-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\tau^2}\,d\tau\,,$$

$$= \frac{\sqrt{2\pi}\,\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\,e^{-x^2/2(\sigma_1^2+\sigma_2^2)}\,,$$

$$h(x,y) = \frac{1}{2\pi(\sigma_1^2+\sigma_2^2)}\exp\left\{\frac{-(x^2+y^2)}{2(\sigma_1^2+\sigma_2^2)}\right\}$$

where,

$$\tau \rightarrow \xi - [\sigma_2^2/(\sigma_1^2+\sigma_2^2)]\,x$$

$$\int_{-\infty}^{\infty} e^{-a^2\tau^2}\,d\tau = \frac{\sqrt{\pi}}{a}\,, \quad \text{for } a \neq 0$$

Separable Systems

2D LSI system with PSF *h(x,y)* is separable if there exist two 1D systems with PSF $h_1(x)$ and $h_2(x)$

$$h(x,y) = h_1(x)h_2(y)$$

A 2D convolution

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi,y-\eta)\,d\xi\,d\eta$$

can be computed as two 1D convolution

$$\text{Compute } w(x,y) = \int_{-\infty}^{\infty} f(\xi,y)h_1(x-\xi)\,d\xi, \text{ for every } y.$$

$$\text{Compute } g(x,y) = \int_{-\infty}^{\infty} w(x,\eta)h_2(y-\eta)\,d\eta, \text{ for every } x.$$

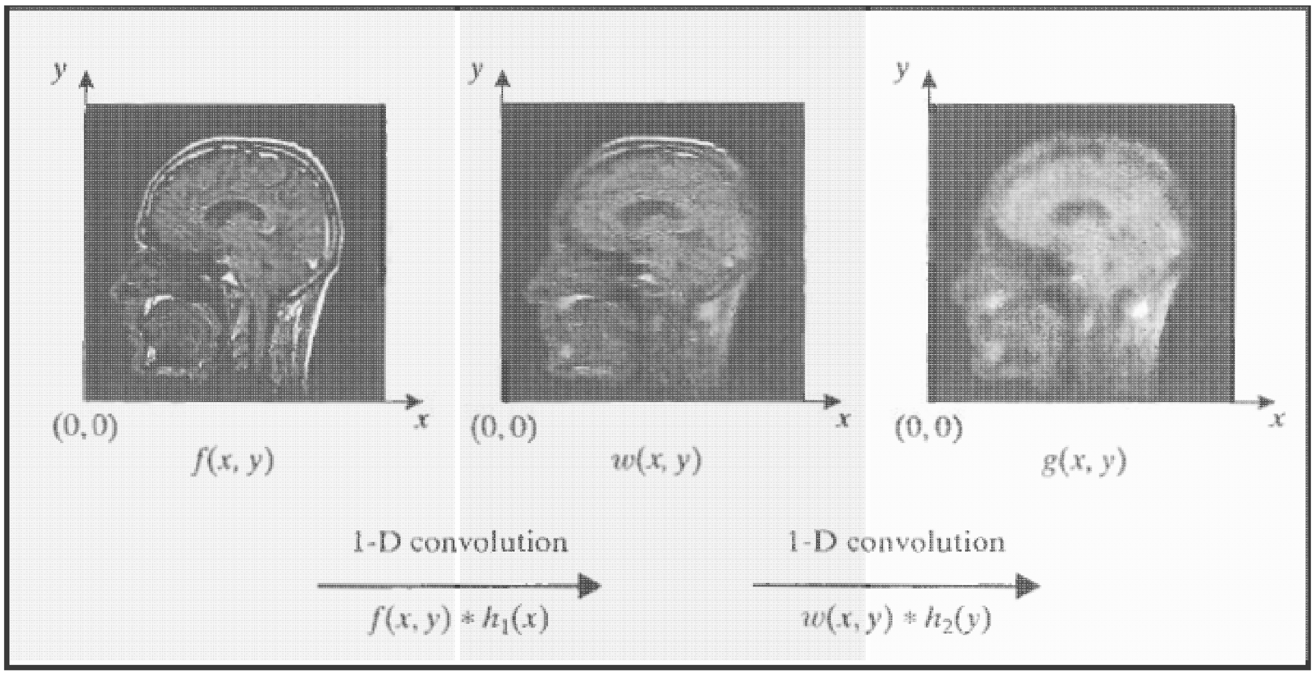
Example 4

The 2 D convolution of the PSF, h(x,y), with an image f(x,y)

$$h(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

Can be calcuated as a two 1D convolution of

$$h_1(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \quad \text{and} \quad h_2(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}$$



Stable Systems

A system is bounded input bounded output stable if and only if when the input is bounded signal

$$|f(x, y)| \leq B < \infty$$

For some finite B, there exists a finite B' such that

$$|g(x, y)| = |h(x, y) * f(x, y)| \leq B' < \infty$$

A LSI system is BIBO if and only if its PSF is absolutely integrable.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(x, y)| \, dx \, dy < \infty$$

The Fourier Transform

The Fourier transform provides a different perspective on how signals and systems interact.
It leads to alternative tools for system analysis and implementation

The 2D Fourier Transform Pair are:

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy ,$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du dv .$$

The Fourier Transform

A sufficient condition for the existence of Fourier Transform:

- f (x,y) is continuous,
- f (x,y) has a finite number of discontinuities,
- f (x,y) is absolutely integrable

$$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)},$$

$$\angle F(u,v) = \tan^{-1}\left(\frac{F_I(u,v)}{F_R(u,v)}\right),$$

$$F(u,v) = |F(u,v)|\, e^{j\angle F(u,v)}.$$

$$F(u,v) = F_R(u,v) + jF_I(u,v)]$$

The Fourier Transform

Example: Find the FT for $\delta(x,y)$

$$\mathcal{F}_{2D}(\delta)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy\,,$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)e^{-j2\pi(u0+v0)}\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)\,dx\,dy = 1$$

Example: Find the FT for $f(x,y)=e^{j2\pi(uox+voy)}$

$$\mathcal{F}_{2D}(f)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy\,,$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy\,,$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u-u_0)x+(v-v_0)y]}\,dx\,dy\,,$$
$$= \delta(u-u_0, v-v_0)\,,$$

where

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)}\,dx\,dy = \delta(u,v)\,,$$

The Fourier Transform

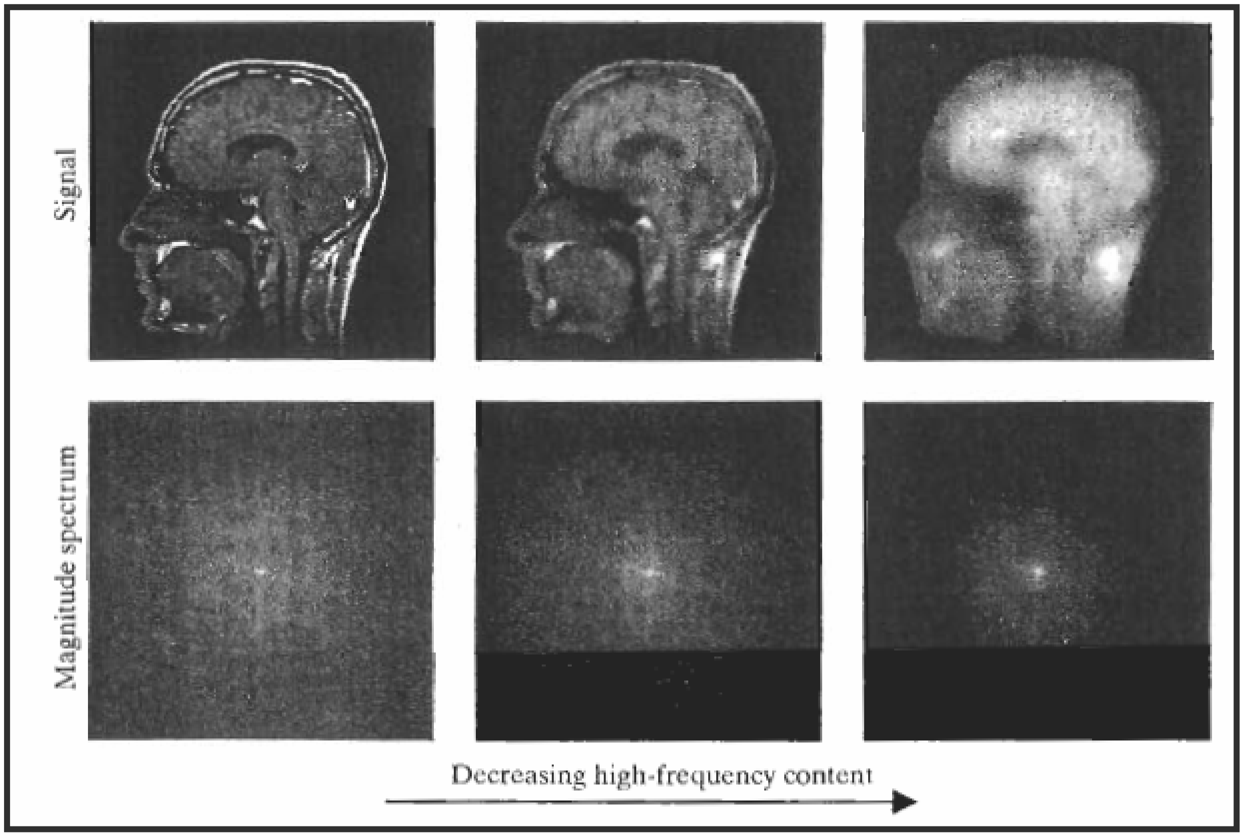
Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\text{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\text{rect}(x, y)$	$\text{sinc}(u, v)$
$\text{sinc}(x, y)$	$\text{rect}(u, v)$
$\text{comb}(x, y)$	$\text{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$

The Fourier Transform

Note:

Slow signal variation in space produces a spectral content that is more concentrated at low frequencies, whereas fast signal variation results in spectral content at high frequencies



The Fourier Transform

The 1D Fourier Transform Pair are:

$$F(u) = \mathcal{F}_{1D}(f)(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx\,,$$

$$f(x) = \mathcal{F}_{1D}^{-1}(F)(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux}\, du\,,$$

Example: Find the FT for rect(x)

$$\begin{aligned}
\mathcal{F}_{1D}(\mathrm{rect})(u) &= \int_{-\infty}^{\infty} \mathrm{rect}(x) e^{-j2\pi ux}\, dx\,,\\
&= \int_{-1/2}^{1/2} e^{-j2\pi ux}\, dx\,,\\
&= \frac{1}{j2\pi u}\, e^{-j2\pi ux}\Big|_{-1/2}^{1/2}\,,\\
&= \frac{1}{\pi u}\,\frac{e^{j\pi u} - e^{-j\pi u}}{2j}\,,\\
&= \frac{\sin(\pi u)}{\pi u} = \mathrm{sinc}(u)\,.
\end{aligned}$$

The Fourier Transform

Properties: Linearity

$$\mathcal{F}_{2D}(a_1 f + a_2 g)(u, v) = a_1 F(u, v) + a_2 G(u, v)$$

Properties: Translation

$$f(x, y) \longrightarrow F(u, v)$$

$$f_{x_0 y_0}(x, y) = f(x - x_0, y - y_0) \longrightarrow \mathcal{F}_{2D}(f_{x_0 y_0})(u, v) = F(u, v) e^{-j2\pi(ux_0 + vy_0)}$$

$$\left|\mathcal{F}_{2D}(f_{x_0 y_0})(u, v)\right| = |F(u, v)|$$

$$\angle\mathcal{F}_{2D}(f_{x_0 y_0})(u, v) = \angle F(u, v) - 2\pi(ux_0 + vy_0)$$

The Fourier Transform

Properties: Conjugation

If *f(x,y)* is complex-valued 2D signal, then

$$\mathcal{F}_{2D}(f^*)(u,v) = F^*(-u,-v)$$

Properties: Conjugate Symmetry

If *f(x,y)* is real-valued 2D signal, then

$$F(u,v) = F^*(-u,-v)\,.$$

$$F_R(u,v) = F_R(-u,-v) \quad \text{and} \quad F_I(u,v) = -F_I(-u,-v)\,,$$

$$|F(u,v)| = |F(-u,-v)| \quad \text{and} \quad \angle F(u,v) = -\angle F(-u,-v)\,.$$

The Fourier Transform

Properties: Scaling

$$f(x, y) \longrightarrow F(u, v)$$

$$f_{ab}(x, y) = f(ax, by) \longrightarrow \mathcal{F}_{2D}(f_{ab})(u, v) = \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

Example: Find the FT for

$$f(x, y) = \text{rect}\left(\frac{x - x_0}{\Delta x}, \frac{y - y_0}{\Delta y}\right)$$

The FT for the rect function is

$$\mathcal{F}_{2D}(\text{rect})(u, v) = \text{sinc}(u, v)$$

Scaling property

$$\mathcal{F}_{2D}\left\{\text{rect}\left(\frac{x}{\Delta x}, \frac{y}{\Delta y}\right)\right\} = \Delta x \Delta y \, \text{sinc}(\Delta x u, \Delta y v)$$

Translation property

$$\mathcal{F}_{2D}(f)(u, v) = \Delta x \Delta y \, \text{sinc}(\Delta x u, \Delta y v) e^{-j2\pi(ux_0 + vy_0)}$$

The Fourier Transform

Properties: Rotation

$$f_\theta(x, y) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

$$\mathcal{F}_{2D}(f_\theta)(u, v) = F(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta)$$

Properties: Convolution

$$\mathcal{F}_{2D}(f * g)(u, v) = F(u, v)G(u, v)$$

The Fourier Transform

Example:

Find the convolution of *f(x,y) = sinc(Ux,Vy)* and *g(x,y)=sinc(Vx,Uy)* where 0<U<V.

The FT for the sinc function is $\mathcal{F}_{2D}\{\text{sinc}(x, y)\} = \text{rect}(u, v)$.

The scaling property

$$F(u, v) = \mathcal{F}_{2D}(f)(u, v) = \frac{1}{UV} \text{rect}\left(\frac{u}{U}, \frac{v}{V}\right) \qquad G(u, v) = \mathcal{F}_{2D}(g)(u, v) = \frac{1}{UV} \text{rect}\left(\frac{u}{V}, \frac{v}{U}\right)$$

The convolution property

$$\begin{aligned} f(x, y) * g(x, y) &= \mathcal{F}_{2D}^{-1}\{F(u, v)G(u, v)\} \\ &= \mathcal{F}_{2D}^{-1}\left\{\frac{1}{(UV)^2} \text{rect}\left(\frac{u}{U}, \frac{v}{V}\right) \text{rect}\left(\frac{u}{V}, \frac{v}{U}\right)\right\} \\ &= \mathcal{F}_{2D}^{-1}\left\{\frac{1}{(UV)^2} \text{rect}\left(\frac{u}{V}, \frac{v}{V}\right)\right\} \\ &= \frac{1}{U^2} \mathcal{F}_{2D}^{-1}\left\{\frac{1}{V^2} \text{rect}\left(\frac{u}{V}, \frac{v}{V}\right)\right\} \\ &= \frac{1}{U^2} \text{sinc}\,(Vx, Vy) \end{aligned}$$

The Fourier Transform

Properties: Product

$$\mathcal{F}_{2D}(fg)(u,v) = F(u,v) * G(u,v)\,,$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)F(u-\xi,v-\eta)\,d\xi\,d\eta$$

Properties: Separable Product

$$f(x,y) = f_1(x)f_2(y)\,,$$

$$\mathcal{F}_{2D}(f)(u,v) = F_1(u)F_2(v)\,,$$

The Fourier Transform

Properties: Parseval's Theorem

The energy contest of a 2D signal in the spatial domain is the same if frequency domain.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\,du\,dv$$

Properties: Separability

The FT of a 2D signal *f(x,y)* can be calculated as two 1D FT

$$r(u,y) = \int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}\,dx, \text{ for every } y.$$

$$F(u,v) = \int_{-\infty}^{\infty} r(u,y)e^{-j2\pi vy}\,dy, \text{ for every } u.$$

The Fourier Transform

Properties of the Fourier transform

Property	Signal	Fourier Transform
	$f(x, y)$	$F(u, v)$
	$g(x, y)$	$G(u, v)$
Linearity	$a_1 f(x, y) + a_2 g(x, y)$	$a_1 F(u, v) + a_2 G(u, v)$
Translation	$f(x - x_0, y - y_0)$	$F(u, v) e^{-j2\pi(ux_0 + vy_0)}$
Conjugation	$f^*(x, y)$	$F^*(-u, -v)$
Conjugate symmetry	$f(x, y)$ is real-valued	$F(u, v) = F^*(-u, -v)$
		$F_R(u, v) = F_R(-u, -v)$
		$F_I(u, v) = -F_I(-u, -v)$
		$\|F(u, v)\| = \|F(-u, -v)\|$
		$\angle F(u, v) = -\angle F(-u, -v)$
Signal reversing	$f(-x, -y)$	$F(-u, -v)$
Scaling	$f(ax, by)$	$\frac{1}{\|ab\|} F\left(\frac{u}{a}, \frac{v}{b}\right)$
Rotation	$f(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$	$F(u \cos\theta - v \sin\theta, u \sin\theta + v \cos\theta)$
Circular symmetry	$f(x, y)$ is circularly symmetric	$F(u, v)$ is circularly symmetric
		$\|F(u, v)\| = F(u, v)$
		$\angle F(u, v) = 0$
Convolution	$f(x, y) * g(x, y)$	$F(u, v) G(u, v)$
Product	$f(x, y) g(x, y)$	$F(u, v) * G(u, v)$
Separable product	$f(x) g(y)$	$F(u) G(v)$
Parseval's theorem	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|f(x, y)\|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|F(u, v)\|^2 \, du dv$.	

The Transfer Function

The transfer function is the FT of the PSF *h(x,y)* and is denoted as *H(u,v).*

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\, d\xi\, d\eta$$

$$h(x,y) = G(u,v) = H(u,v)F(u,v)^{vy)}\, du\, dv$$

The transfer function is the ratio of the output signal to input signal in the frequency domain.

$$G(u,v) = H(u,v)F(u,v)$$

The Transfer Function

Example: Low Pass Filter (LPF)

The transfer function for a LPF is:

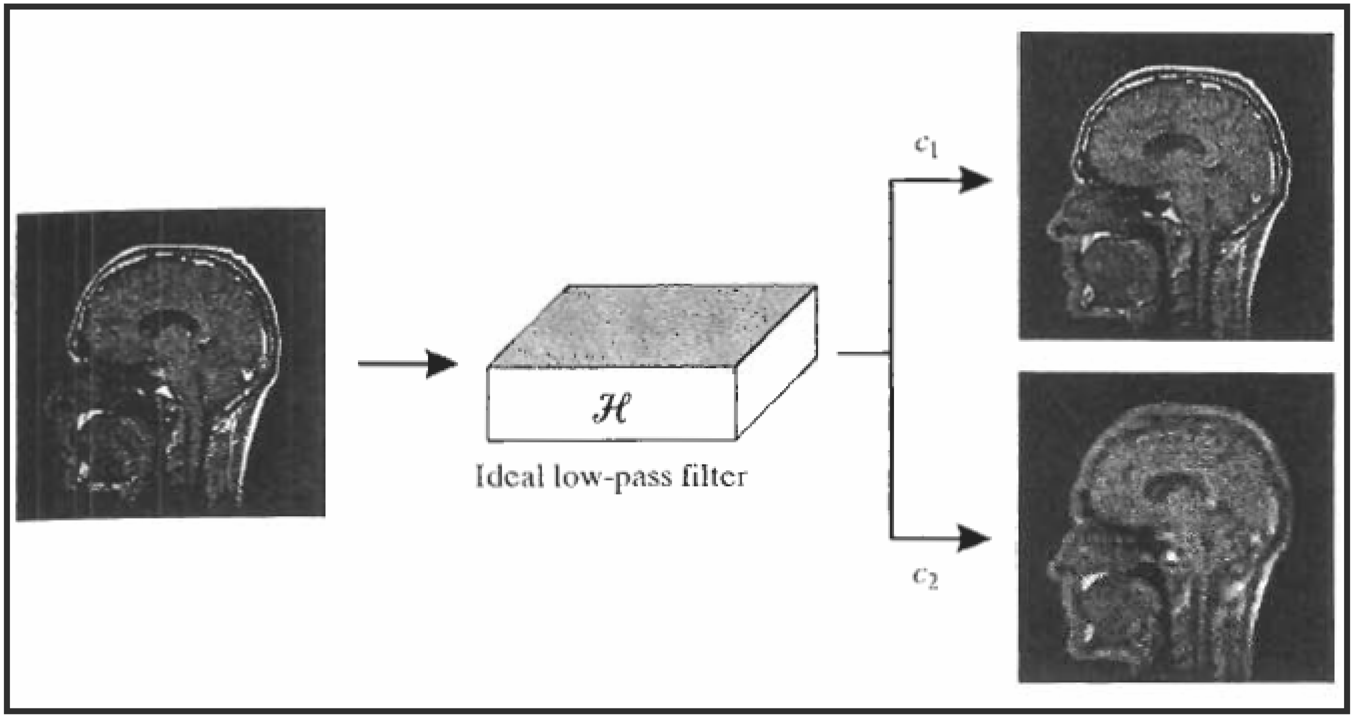
$$H(u,v) = \begin{cases} 1, & \text{for } \sqrt{u^2+v^2} \le c \\ 0, & \text{for } \sqrt{u^2+v^2} > c \end{cases}$$

The input output signals are:

$$G(u,v) = \begin{cases} F(u,v), & \text{for } \sqrt{u^2+v^2} \le c \\ 0, & \text{for } \sqrt{u^2+v^2} > c \end{cases}$$

The Transfer Function

Example, LPF cont.,



Circular Symmetry:

A 2D signal is said to have a circular symmetry if

$$f_\theta(x, y) = f(x, y), \qquad \text{for every } \theta$$

where

$$f_\theta(x, y) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

The circular symmetry implies *f(x,y)* is even, which gives even real *F(u,v)*

$$|F(u, v)| = F(u, v) \quad \text{and} \quad \angle F(u, v) = 0$$

Hankel Transform:

Hankel transform is defined for circularly symmetric signals.

Define $r = \sqrt{x^2 + y^2}$ Then $f(x, y) = \mathrm{f}(r)$,

And $F(u, v) = \mathrm{F}(q)$ For $q = \sqrt{u^2 + v^2}$.

Hankel Transform is

$$\mathrm{F}(q) = \mathcal{H}\{\mathrm{f}(r)\}$$

$$\mathrm{F}(q) = 2\pi \int_0^\infty \mathrm{f}(r) J_0(2\pi q r)\, r\, dr$$

$$J_n(r) = \frac{1}{\pi} \int_0^\pi \cos(nr - r\sin\phi)\, d\phi, \quad n = 0, 1, 2, \ldots,$$

$$J_0(r) = \frac{1}{\pi} \int_0^\pi \cos(r\sin\phi)\, d\phi$$

Hankel Inverse transform is

$$\mathrm{f}(r) = 2\pi \int_0^\infty \mathrm{F}(q) J_0(2\pi q r)\, q\, dq$$

Hankel Transform:

Selected Hankel Transform Pairs

Signal	Hankel Transform
$\exp\{-\pi r^2\}$	$\exp\{-\pi q^2\}$
1	$\delta(q)/\pi q = \delta(u, v)$
$\delta(r - a)$	$2\pi a J_0(2\pi a q)$
$\mathrm{rect}(r)$	$\frac{J_1(\pi q)}{2q}$
$\mathrm{sinc}(r)$	$\frac{2\,\mathrm{rect}(q)}{\pi\sqrt{1-4q^2}}$
$\frac{1}{r}$	$\frac{1}{q}$

Properties: Scaling (a=b)

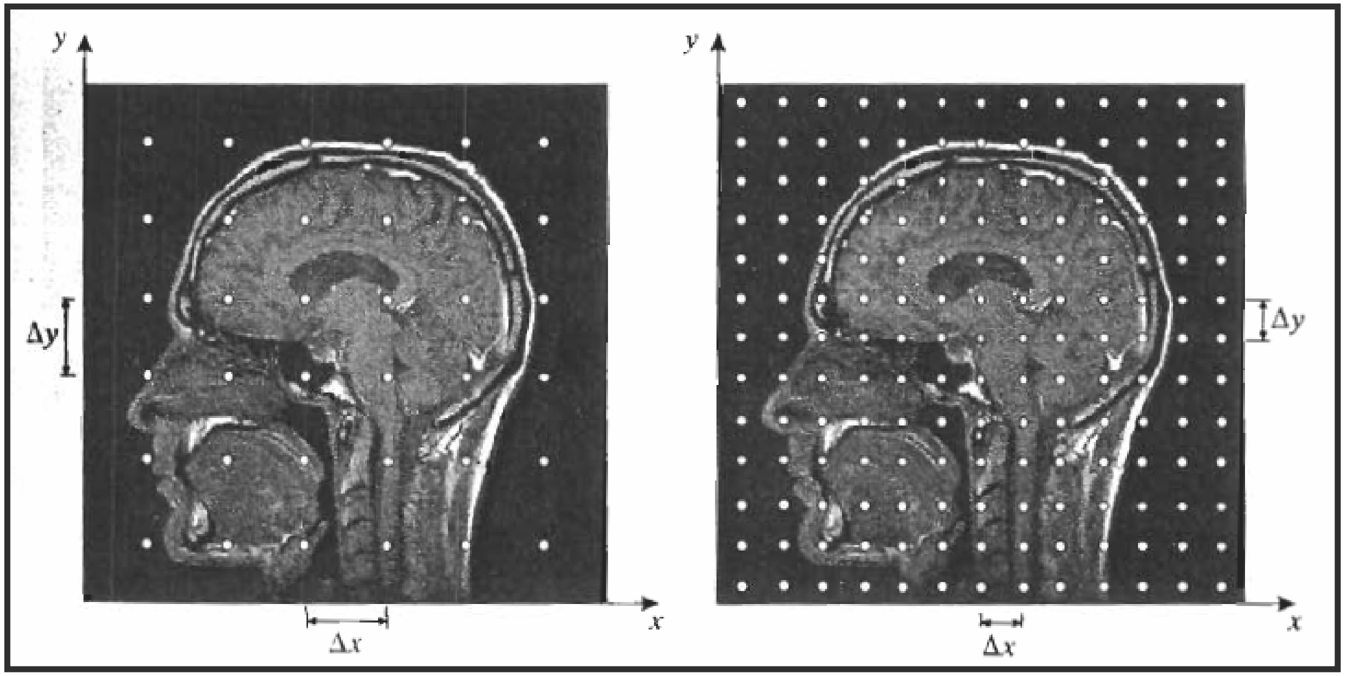
$$\mathcal{H}\{f(ar)\} = \frac{1}{a^2}F(q/a)$$

Sampling:

A 2D continuous signal is replaced by a discrete signal whose values are the values of the continuous signal at the vertices of a 2D rectangular grid.

For Δx and Δy sampling periods

$$f_d(m, n) = f(m\Delta x, n\Delta y)$$



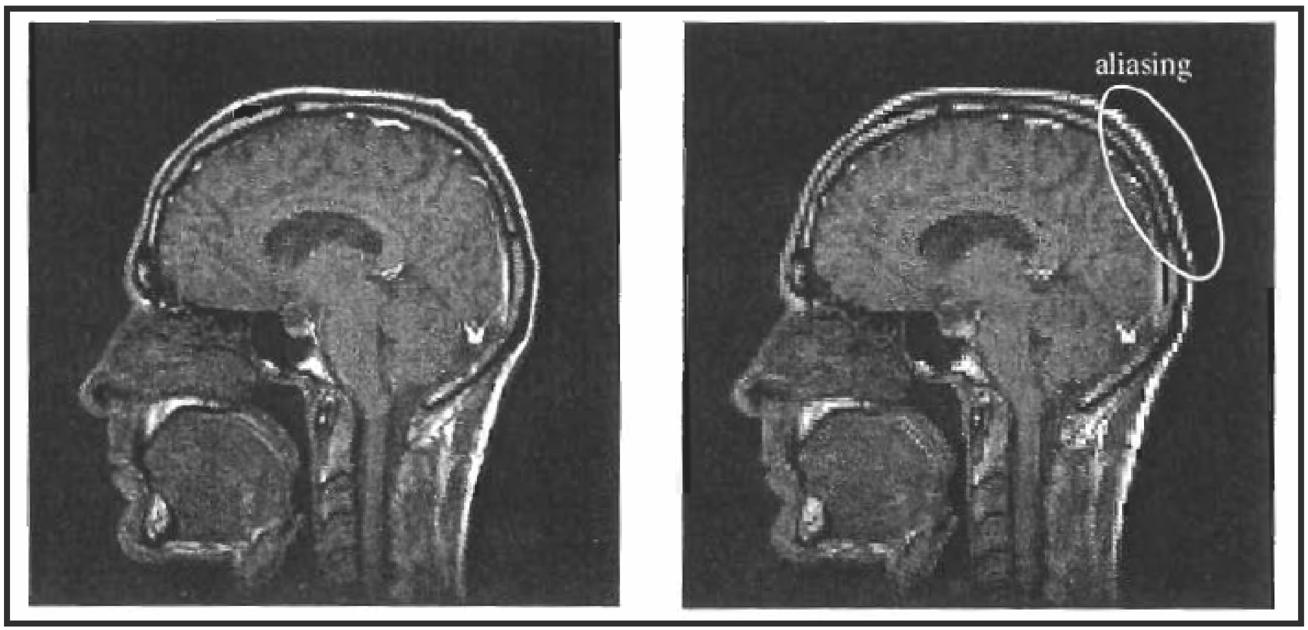
Sampling:

More sampling:

More detectors, signal storage, subsequent processing, etc.

Less sampling:

Aliasing



Sampling:

The sampled 2D signal is spatial domain is:

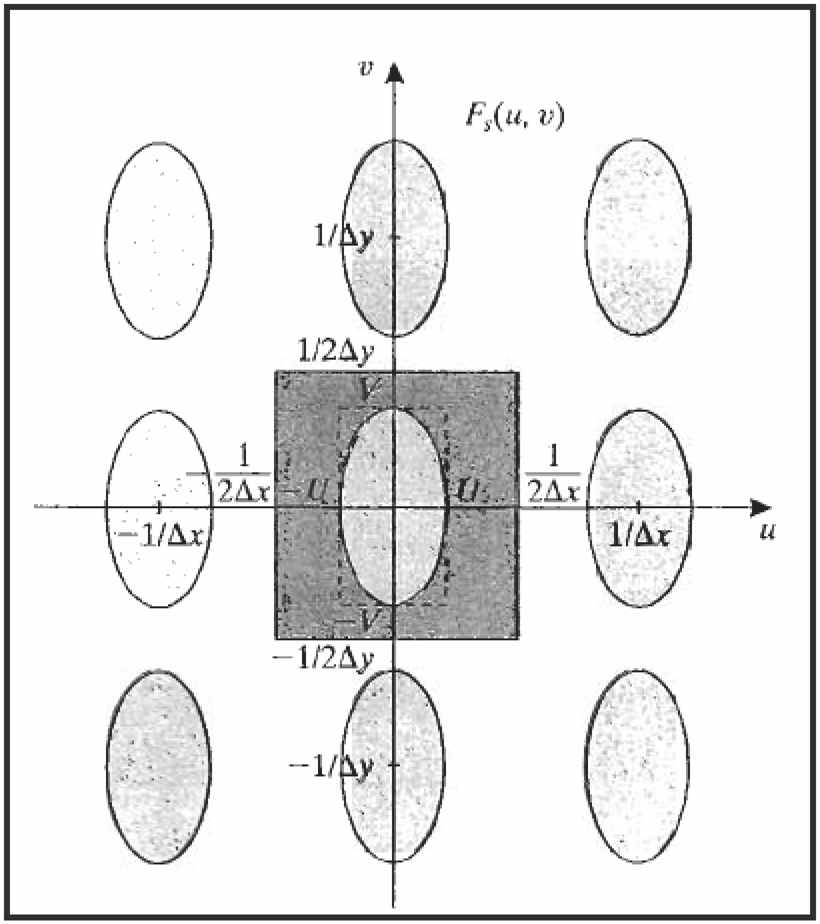
$$f_s(x,y) = f(x,y)\delta_s(x,y;\Delta x,\Delta y)$$
$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f_d(m,n)\delta(x - m\Delta x, y - n\Delta y)$$

The sampled 2D signal in frequency domain is:

$$F_s(u,v) = F(u,v) * \mathrm{comb}(u\Delta x, v\Delta y)$$
$$= \frac{1}{\Delta x \Delta y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} F(u - m/\Delta x, v - n/\Delta y)$$

Sampling:

The spectrum of the sampled 2D signal is:

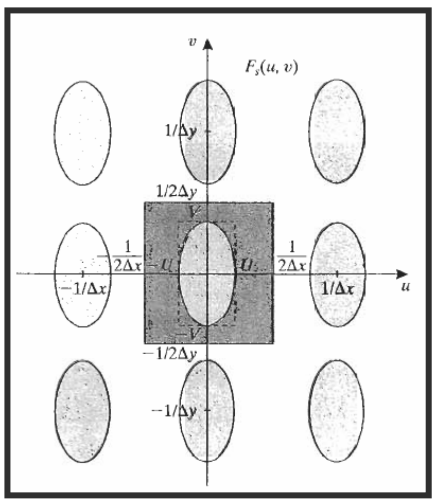


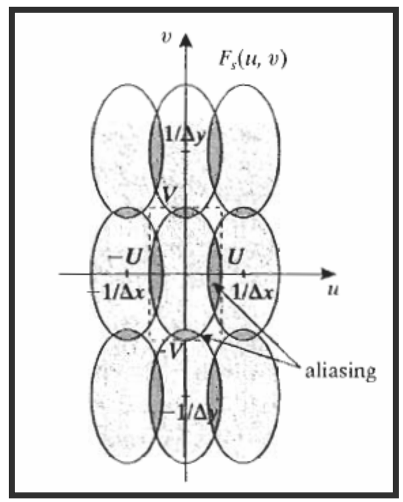
Sampling:

Nyquist Sampling Theorem

A 2D continuous band-limited signal *f(x,y)* with cutoff frequencies U and V can be UNIQUELY determined from its samples $f_d(m,n)$ if and only if the sampling periods Δx and Δy satisfy:

$$\Delta x \le \frac{1}{2U} \quad \text{and} \quad \Delta y \le \frac{1}{2V}$$





Thanks