Classical Mechanics 95.611

Homework # 1 (Due to Feb. 03, 2015).
Chapter 1. Survey of the Elementary Principles

Problem A. **(10 points)**

A Yo-Yo (mass m and radius R) suspended from the ceiling by a string of negligible mass. Find the acceleration of the center of the Yo-Yo and the tension of the string.

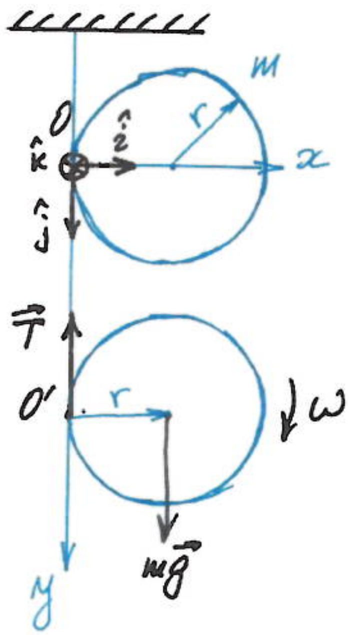
Problem 1.8. **(10 points)**

Problem 1.11(a). **(10 points)**

Problem 1.18. **(10 points)**

Problem 1.20. **(10 points)**

Problem A A Yo-Yo suspended from the ceiling by a string of negligible mass. Find the acceleration of the center of the Yo-Yo and the tension of the string.



Suppose that initially the disc has a point O in contact with a string and that after time t the disc has rotated through angle θ.

The forces acting on the disc are

1. the weight $m\vec{g}$
2. tension $\vec{T}$

- If $\vec{r}$ is the position of the center of mass, then by Newton's 2nd law

$$m\ddot{\vec{r}} = m\vec{g} - \vec{T}$$

$$m(\hat{i}\underbrace{\ddot{x}}_{0} + \hat{j}\ddot{y}) = mg\cdot\hat{j} - T\cdot\hat{j}$$

$$m\ddot{y} = mg - T \qquad (1)$$

- The total angular momentum about the horiz. axis through the CM.

$$\vec{L} = I_c\cdot\vec{\omega} = I_c\cdot\dot{\theta}\hat{k} = \left\| \begin{matrix}\text{moment of inertia}\\ I_c = \frac{1}{2}mr^2\end{matrix} \right\| = \frac{1}{2}mr^2\dot{\theta}\cdot\hat{k}$$

- Now, the torque about the CM:

$$\vec{\tau} = \Sigma\vec{\tau}_i = 0\times m\vec{g} + r(-\hat{i})\times T(-\hat{j}) = \left\| \hat{i}\times\hat{j} = \hat{k} \right\| = rT\cdot\hat{k}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \underline{rT\hat{k}} = \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}\hat{k}\right) = \frac{1}{2}\underline{mr^2\ddot{\theta}}\hat{k}$$

$$T = \frac{mr}{2}\cdot\ddot{\theta}$$

If there is no slipping, $y = \theta\cdot r \Rightarrow \theta = y/r \Rightarrow \ddot{\theta} = \ddot{y}/r$

$$T = \frac{mr}{2}\ddot{\theta} = \frac{mr}{2}\cdot\frac{\ddot{y}}{r} = \frac{m}{2}\ddot{y} \qquad (2)$$

now we can combine (1) and (2)

$$m\ddot{y} = mg - T = mg - \frac{m}{2}\ddot{y}$$

$$\frac{3}{2}m\ddot{y} = mg$$

$$\boxed{\ddot{y} = \frac{2}{3}g} \quad \therefore$$

now, this can be used in (2)

$$T = \frac{m}{2}\ddot{y} = \frac{m}{2}\cdot\frac{2}{3}g = \boxed{\frac{mg}{3} = T} \quad \therefore$$

1.1 Show that for a particle with constant mass the eq-n of motion implies the following DE for the KE:

$$\frac{dT}{dt} = \vec{F}\cdot\vec{v}$$

while if the mass varies with time the corresponding eq-n is $$\frac{d(mT)}{dt} = \vec{F}\cdot\vec{p}$$

- $T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v}$ – the KE of the particle

$$\frac{dT}{dt} = \frac{1}{2}m\left[\left(\frac{d\vec{v}}{dt}\cdot\vec{v}\right)+\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right)\right] = m\left(\frac{d\vec{v}}{dt}\cdot\vec{v}\right) = \left\| \begin{matrix}\text{since}\\ \underline{m=\text{const}}\end{matrix}\right\| = \frac{d(m\vec{v})}{dt}\cdot\vec{v} =$$
$$= \frac{d\vec{p}}{dt}\cdot\vec{v} = \vec{F}\cdot\vec{v} \quad \therefore$$

- $T\cdot m = \frac{m^2v^2}{2} = \frac{m\vec{v}\cdot m\vec{v}}{2} = \frac{\vec{p}\cdot\vec{p}}{2}$

$$\frac{d(T\cdot m)}{dt} = \frac{1}{2}\left(\frac{d\vec{p}}{dt}\cdot\vec{p}+\vec{p}\cdot\frac{d\vec{p}}{dt}\right) = \frac{d\vec{p}}{dt}\cdot\vec{p} = \left\|\vec{F}=\frac{d\vec{p}}{dt}\right\| = \vec{F}\cdot\vec{p}$$

here we did not use $m=$ const.
It's in $\frac{d\vec{p}}{dt}$

1.8 If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's eq-ns, show by direct substitution that

$$L' = L + \frac{dF(q_1,\dots,q_n,t)}{dt}$$

also satisfies Lagrange's eq-ns where F is any arbitrary, but differentiable, function of its arguments.

Given $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

- $\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = 0 \Leftarrow L' = L + dF/dt$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(L+\frac{dF}{dt}\right)-\frac{\partial}{\partial q_i}\left(L+\frac{dF}{dt}\right)=0 \qquad \frac{dF}{dt}=\dot{F}$$

$$\underbrace{\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}\right]}_{=0\text{ (given)}}+\frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}_i}-\frac{\partial \dot{F}}{\partial q_i}=0$$

$$(*) \qquad \frac{d}{dt}\frac{\partial \dot{F}}{\partial \dot{q}_i}-\frac{\partial \dot{F}}{\partial q_i}=0 \quad \Leftarrow \text{now this has to be proven.}$$

$$\dot{F}\equiv\frac{dF(q_j,t)}{dt}=\sum_j\frac{\partial F}{\partial q_j}\cdot\frac{\partial q_j}{\partial t}+\frac{\partial F}{\partial t}=\left\|\frac{\partial q_j}{\partial t}=\frac{dq_j}{dt}=\dot{q}_j\right\|=\sum_j\frac{\partial F}{\partial q_j}\dot{q}_j+\frac{\partial F}{\partial t}$$

$$\frac{\partial}{\partial \dot{q}_i}\left(\frac{dF}{dt}\right)=\frac{\partial}{\partial \dot{q}_i}\left(\sum_j\frac{\partial F}{\partial q_j}\dot{q}_j+\frac{\partial F}{\partial t}\right)=\frac{\partial F}{\partial q_i} \quad \text{now this can be used in } (*)$$

$$\frac{d}{dt}\frac{\partial F}{\partial q_i}-\frac{\partial \dot{F}}{\partial q_i}=0$$

Since order of differentiation can be changed,

$$\frac{\partial}{\partial q_i}\frac{dF}{dt}-\frac{\partial \dot{F}}{\partial q_i}=\frac{\partial \dot{F}}{\partial q_i}-\frac{\partial \dot{F}}{\partial q_i}=0.$$

So $(*)$ is true and, as a result, L' satisfies Lagrange's eq-ns

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i}-\frac{\partial L'}{\partial q_i}=0.$$

1.11

11. Consider a uniform thin disk that rolls without slipping on a horizontal plane. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk.

(a) Derive Lagrange's equations and find the generalized force.

(b) Discuss the motion if the force is not applied parallel to the plane of the disk.

a) The generalized force can be related to the KE by the eq-n (1.53).

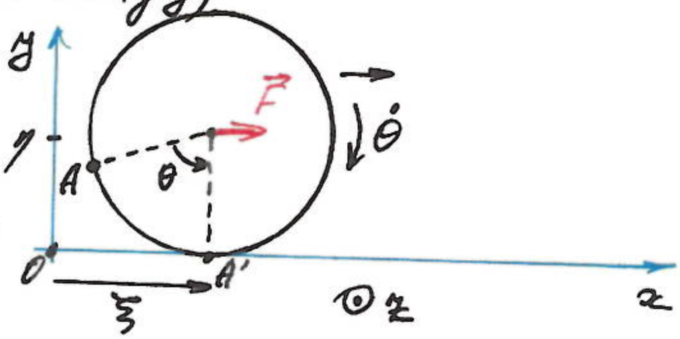
(1) $$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

So, we need to choose the generalized coordinates and find an expression for T (kinetic energy)

The coord of the CM are the generalized coord. (ξ, η)

$\eta = R =$ const (radius of the disk)

Introduce also, θ, the angle of rotation of the disk. At $t=0$ point A was at the origin O.



So, no slipping condition is $OA' = AA' \Rightarrow R\cdot\theta = \xi$

or $\dot{\xi} = R\cdot\dot{\theta}$

The KE of the disk is

$$T = \frac{M}{2}\cdot V_{CM}^2 + \frac{I}{2}\cdot\omega^2 = \left\| \begin{array}{l} V_{CM}^2 = \dot{\xi}^2 + \overset{0}{\dot{\eta}^2} = \dot{\xi}^2 \\ \omega = \dot{\theta} \\ I = \frac{1}{2}MR^2 \text{ (moment of inertia)} \end{array} \right\| = \frac{M}{2}\dot{\xi}^2 + \frac{M}{4}R^2\dot{\theta}^2 =$$

$$= \| \dot{\xi} = R\dot{\theta} \| = \frac{M}{2}\dot{\xi}^2 + \frac{M}{4}\cancel{R^2}\cdot\frac{\dot{\xi}^2}{\cancel{R^2}} = \frac{3}{4}M\cdot\dot{\xi}^2$$

Substitute it into (1)

$$\frac{d}{dt}\left[\frac{\partial}{\partial\dot{\xi}}\left(\frac{3}{4}M\dot{\xi}^2\right)\right] - \frac{\partial}{\partial\xi}\left(\frac{3}{4}M\dot{\xi}^2\right) = Q$$

$$Q = \frac{d}{dt}\left(\frac{3}{2}M\dot{\xi}\right) = \boxed{\frac{3}{2}M\ddot{\xi} = Q}$$

1.18

18. A Lagrangian for a particular physical system can be written as

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{K}{2}\left(ax^2 + 2bxy + cy^2\right),$$

where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and $b = 0$, $c = -a$. What is the physical system described by the above Lagrangian? Show that the usual Lagrangian for this system as defined by Eq. (1.56) is related to L' by a point transformation (cf. Derivation 10). What is the significance of the condition on the value of $b^2 - ac$?

• First, let's find the eq-ns of motion:

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} - \frac{\partial L'}{\partial x} = 0 \qquad\qquad \frac{d}{dt}\frac{\partial L'}{\partial \dot{y}} - \frac{\partial L'}{\partial y} = 0$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} = \frac{m}{2}\frac{d}{dt}\left(2a\dot{x} + 2b\dot{y}\right) = 2(a\ddot{x} + b\ddot{y})\cdot\frac{m}{2} \qquad \frac{d}{dt}\frac{\partial L'}{\partial \dot{y}} = m(c\ddot{y} + b\ddot{x})$$

$$\frac{\partial L'}{\partial x} = -\frac{K}{2}\cdot(2ax + 2by) = -K(ax + by) \qquad \frac{\partial L'}{\partial y} = -K\cdot(cy + bx)$$

$$(1)\quad m(a\ddot{x} + b\ddot{y}) = -K(ax + by) \qquad\qquad m(b\ddot{x} + c\ddot{y}) = -K(bx + cy) \quad (2)$$

$$(3)\quad \begin{cases} m(a\ddot{x} + b\ddot{y}) = -K(ax + by) \\ m(b\ddot{x} + c\ddot{y}) = -K(bx + cy) \end{cases}$$

These are the eq-ns of motion of a 2D harmonical oscillator of mass m constrained to move in the xy plane and pulled to the origin by two springs K.

• Examine particularly the two cases $\underline{a = 0 = c}$

$$\begin{cases} m\ddot{y} = -Ky \\ m\ddot{x} = -Kx \end{cases}$$

$b = 0$, $c = -a$

$$\begin{cases} m\ddot{x} = -Kx \\ m\ddot{y} = -Ky \end{cases}$$

Set of eq-ns (3) can be written in matrix form

$$(4)\quad m\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = -K\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

so we have coupled DE. They can be decoupled by going to a different coord. system $\begin{array}{l} \xi = ax + by \\ \eta = bx + cy \end{array}$ (this is a point transform) (see problem 1.10

by rotating the coord. system. It can be done by multiplying both sides of the eq-us (4) by the inverse matrix of the matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

The inverse is $A^{-1} = \frac{1}{\|A\|} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \frac{1}{(ac-b^2)} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$

matrix of cofactors.

The inverse matrix (A^{-1}) exists (and, as a result, the rotation is possible) when $\det A \neq 0 \Rightarrow ac - b^2 \neq 0$.

Then, the eq-us of motion are

$$\begin{cases} m\ddot{\xi} = -k\xi \\ m\ddot{\eta} = -k\eta \end{cases}$$

So, now we have two independent equations, which corresponds to the usual Lagrangian (KE-PE):

$$L = T - V = \frac{m}{2}\left(\dot{\xi}^2 + \dot{\eta}^2\right) - \frac{k}{2}\left(\xi^2 + \eta^2\right)$$

If $b^2 - ac = 0$, then $b = \sqrt{ac}$, and

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2\sqrt{ac}\,\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{k}{2}\left(ax^2 + 2\sqrt{ac}\,xy + cy^2\right)$$

$$L' = \frac{m}{2}\left(\sqrt{a}\,\dot{x} + \sqrt{c}\,\dot{y}\right)^2 - \frac{k}{2}\left(\sqrt{a}\,x + \sqrt{c}\,y\right)^2 = \left\| \sqrt{a}\,x + \sqrt{c}\,y = u \right\| = \frac{m}{2}\dot{u}^2 - \frac{k}{2}u^2$$

so the condition $b^2 - ac = 0$ reduces the system to a one dimensional.

(1.20) A particle of mass m moves in one dimension such that it has the Lagrangian.

$$L = \frac{m^2 \dot{x}^4}{12} + m \cdot \dot{x}^2 \cdot V(x) - V^2(x)$$

where V is some differentiable f-n of x. Find the eq-n of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

Based on L, units of $V(x)$ are Joules (Energy)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{4m^2\dot{x}^3}{12} + 2m\dot{x}V(x)\right) - \left(m\dot{x}^2\frac{dV}{dx} - 2V\frac{dV}{dx}\right) = 0$$

$$\frac{m^2}{3}\cdot 3\dot{x}^2\cdot\ddot{x} + 2m\ddot{x}\cdot V(x) + 2m\dot{x}\frac{dV}{dx}\dot{x} - m\dot{x}^2\frac{dV}{dx} + 2V\frac{dV}{dx} = 0$$

$$\underbrace{\left(m\dot{x}^2 + 2V(x)\right)}_{\text{units of energy}} \cdot \underbrace{\left(m\ddot{x} + \frac{dV}{dx}\right)}_{\text{units of force.}} = 0$$

so, there are two options:

- $m\dot{x}^2 = -2V(x)$

  $\dot{x} = \sqrt{-\frac{2V(x)}{m}}$, $V < 0$ (must be)

- $m\ddot{x} = -\frac{dV(x)}{dx}$

Further integration is not possible without knowing $V(x)$