92.530 Applied Mathematics I: Solutions to Homework Problems in Chapter 7

- **7.26 (c)** If you begin by subtracting 20 from the function, it becomes an odd function, leading to a sine series.

- **7.26(d)** Since the period is given as $2L = 6$, we seek a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{3}x) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{3}x).$$

Using integration by parts once, we calculate

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos(\frac{n\pi}{3})dx = \frac{1}{3}\int_{0}^{3} 2x\cos(\frac{n\pi}{3})dx$$

and find that

$$a_n = 6\frac{(\cos(n\pi) - 1)}{(n\pi)^2}.$$

Also

$$a_0 = \frac{1}{3}\int_0^3 2x dx = 3.$$

Similarly,

$$b_n = \frac{1}{3}\int_0^{\infty} 2x\sin(\frac{n\pi}{3}x) = -6\frac{\cos(n\pi)}{n\pi}.$$

- **7.27** In part (a), the discontinuities are at $x = 2$, and again at every point having the form $x = 2 + 2m$, for any integer m. The Fourier series converges to 0, the mean of 8 and -8, at these discontinuity points. In part (b), $f(x)$ has no discontinuities. In part (c), $f(x)$ has a discontinuity at $x = 0$, and again at every point of the form $x = 10m$, for any integer m. The Fourier series converges, at these points of discontinuity, to the value 20. Finally, in part (d), $f(x)$ has a discontinuity at $x = 3$ and again at every point of the form $x = 3 + 6m$, for any integer m. The Fourier series converges, at these points of discontinuity, to the value 3.

- **7.29** When we extend $f(x)$ periodically, with period $2L = \pi$, we get an odd function. Therefore, its Fourier series is automatically a sine series,

$$\sum_{n=1}^{\infty} b_n \sin(2nx),$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(2nx).$$

Using integration by parts twice, we find that

$$\int_0^{\pi} \cos(x) \sin(2nx) dx = \frac{4n}{4n^2 - 1}.$$

- **7.30** When we extend $f(x)$ so that the period is $2L = \pi$, the resulting function is odd. Therefore, its Fourier series is just a sine series, and, in fact, is the same series we obtained in the previous exercise.

- **7.42** We need to find constants a_0, a_1, a_2, a_3, a_4, and a_5 so that

$$\int_{-1}^{1} a_0(a_1 + a_2 x) dx = 0,$$

$$\int_{-1}^{1} a_0(a_3 + a_4 x + a_5 x^2) dx = 0,$$

and

$$\int_{-1}^{1} (a_1 + a_2 x)(a_3 + a_4 x + a_5 x^2) dx = 0.$$

Notice that we can assume, for simplicity, that $a_0 = a_2 = a_5 = 1$, and then normalize at the end. We want

$$0 = \int_{-1}^{1} (a_1 + x) dx = a_1 x|_{-1}^{1} + x^2|_{-1}^{1} = 2a_1 + 0,$$

so $a_1 = 0$. Also, we want

$$0 = \int_{-1}^{1} (a_3 + a_4 x + x^2) dx = 2a_3 + \frac{2}{3},$$

so $a_3 = -\frac{1}{3}$. Finally, we want

$$0 = \int_{-1}^{1} x(\frac{-1}{3} + a_4 x + x^2) dx = \frac{2}{3} a_4,$$

so $a_4 = 0$. The three orthogonal polynomials are then 1, x, and $x^2 - \frac{1}{3}$, or any scalar multiples of these. Now we normalize, to get an orthonormal set.

The first polynomial is a constant, $P_1(x) = c$, with

$$\int_{-1}^{1} c^2 dx = 1.$$

It follows that $c = \frac{1}{\sqrt{2}}$, so that

$$P_1(x) = \frac{1}{\sqrt{2}}.$$

We have

$$\int_{-1}^{1} x^2 dx = \frac{2}{3},$$

so the second polynomial is

$$P_2(x) = \sqrt{\frac{3}{2}}x.$$

Finally,

$$\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \frac{8}{45},$$

so that the third polynomial in the orthonormal family is

$$P_3(x) = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}).$$

- **7.46** Suppose that $r = (a, b, c)$. Then $a = r \cdot i$ and $b = r \cdot j$, so

$$(r \cdot i)^2 + (r \cdot j)^2 = a^2 + b^2 \leq a^2 + b^2 + c^2 = |r|^2.$$

- **7.48** We multiply out

$$F(c_1, ..., c_M) = \int_a^b \Big[f(x) - \sum_{n=1}^{M} c_n \phi_n(x)\Big]^2 dx$$

and use the fact that

$$\int_a^b \phi_n(x)\phi_m(x)dx = 0,$$

if m and n are not the same, and equals one if they are, to get

$$F(c_1, ..., c_M) = \int_a^b f(x)^2 dx - 2\sum_{n=1}^{M} c_n \int_a^b f(x)\phi_n(x)dx + \sum_{n=1}^{M} c_n^2.$$

Since this is a function of the M variables c_1,...,c_M, we set to zero the partial derivatives of this function with respect to each of the c_n. Then we have

$$0 = -2\int_a^b f(x)\phi_n(x)dx + 2c_n,$$

so that

$$c_n = \int_a^b f(x)\phi_n(x).$$

- **7.49** Using integration by parts to obtain the recursion

$$\int_0^\infty x^n e^{-x} dx = n \int_0^\infty x^{n-1} e^{-x} dx,$$

and use it to show that

$$\int_0^\infty x^n e^{-x} dx = n!,$$

for $n = 0, 1, ...$ Now we show, for example, that

$$0 = \int_0^\infty (1 - x)(2 - 4x + x^2) e^{-x} dx.$$

This becomes

$$0 = \int_0^\infty 2e^{-x} - 6xe^{-x} + 5x^2 e^{-x} - x^3 e^{-x} dx,$$

or

$$0 = 2(0!) - 6(1!) + 5(2!) - 1(3!) = 2 - 6 + 10 - 6,$$

which is true. The other calculations are similar.