CHAPTER 14

Traffic Flow Theory

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1. Basic Equations

Let us derive the basic conservation equation for traffic flow. We consider the flow of vehicles on a long road where the features of the flow we wish to calculate, such as bottlenecks, etc., are long compared with the average distances between vehicles. Let $n(x, x + \Delta x, t)$ denote the number of vehicles between point x and point $x + \Delta x$ on the road at time t (see Figure 14.1). We shall *assume* that $k(x, t)$ exists such that for any x, Δx, and t,

$$n(x, x + \Delta x, t) = \int_{x}^{x+\Delta x} k(\hat{x}, t)\, d\hat{x}. \tag{1}$$

We note that, by the fundamental theorem of calculus,

$$k(x, t) = \lim_{\Delta x \to 0} \frac{n(x, x + \Delta x, t)}{\Delta x}$$

if k is continuous. We shall assume that we can adequately model the situations of interest with the assumption that k is continuous.

In terms of infinitesimals, k is the number of vehicles per unit length in the infinitesimal length between x and $x + \Delta x$ at time t. Empirical values of k can be determined from aerial photographs of roads. We select some "small" (infinitesimal) length Δx, count the vehicles between x and $x + \Delta x$, and divide by Δx.

Now let us define the flow rate $q(x, t)$. The flow rate q is simply the rate at which vehicles pass point x at time t. The total number Q crossing point x between time t and time $t + \Delta t$ is then given by

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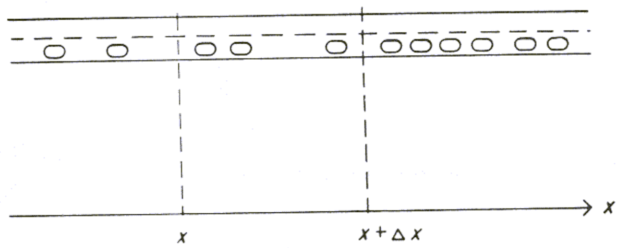

Figure 14.1. Traffic Situation at Some Time t

$$Q(x, t, t + \Delta t) = \int_t^{t+\Delta t} q(x, \hat{t})\, d\hat{t}. \tag{2}$$

Again, by the fundamental theorem of calculus, we have

$$q(x, t) = \lim_{\Delta t \to 0} \frac{Q(x, t, t + \Delta t)}{\Delta t}. \tag{3}$$

Empirical values of q can be obtained by clocked counters which keep a time record of the vehicles crossing point x. We select some "small" Δt, count the vehicles crossing x between t and $t + \Delta t$, and divide by Δt.

Let us now consider the balance or conservation of vehicles in the road. Let us isolate a segment of the road lying between points x and $x + \Delta x$ and look at the rate of change of the number of vehicles in this segment.

The balance law which applies here is that no vehicles are created or destroyed (neglecting collisions!) in this segment. Thus conservation of vehicles requires that *the rate of increase of the number of vehicles between* x *and* $x + \Delta x$ *is equal to the rate at which vehicles flow in minus the rate at which they flow out.* Thus for any time instant t

$$\frac{d}{dt} \int_x^{x+\Delta x} k(\hat{x}, t)\, d\hat{x} = q(x, t) - q(x + \Delta x, t). \tag{4}$$

This is the fundamental conservation law (balance law) for the segment of road between x and $x + \Delta x$; it is a statement about the balance we would see in a snapshot of the road (see Figure 14.1). We note that

$$\frac{d}{dt} \int_x^{x+\Delta x} k(\hat{x}, t)\, d\hat{x} = \int_x^{x+\Delta x} \frac{\partial k}{\partial t}(\hat{x}, t)\, d\hat{x}. \tag{5}$$

Let us now divide by Δx and let $\Delta x \to 0$. We have

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} \frac{\partial k}{\partial t}(\hat{x}, t)\, d\hat{x} = \lim_{\Delta x \to 0} \frac{q(x, t) - q(x + \Delta x, t)}{\Delta x}. \tag{6}$$

By the fundamental theorem of calculus, the limit on the left is precisely $(\partial k/\partial t)(x, t)$, while by the definition of partial derivative, the limit of the quotient on the right is $-(\partial q/\partial x)(x, t)$. Thus we arrive at the fundamental

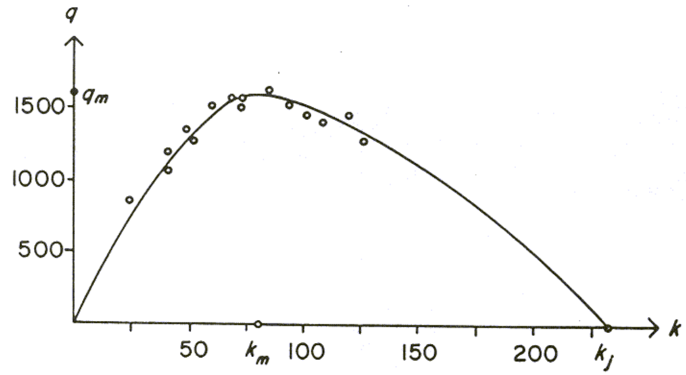

Figure 14.2. Flow Versus Concentration

balance law in differential form

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{7}$$

This equation tells us how the concentration k changes in time at each x from the flow q. To predict how k changes, then, knowledge of another variable q is required.

To see that this equation is qualitatively correct, consider the following thought experiment. If $\partial q/\partial x < 0$, then q is decreasing in a neighborhood of the location x, and $q(x + \varepsilon, t)$ is less than $q(x - \varepsilon, t)$ for some small positive number ε. Thus the flow out of this section of road is less than the flow in, and hence k must increase in time near that location. Mathematically, this is expressed by (7).

We note that we have one equation (7) for two unknown functions k and q. Thus our system is *underdetermined.* A little more thought should convince us that this underdetermination is necessary at this stage. After all, we have in no essential way used the fact that we are modeling *vehicular traffic*. The concentration and flow could just as well be concentration and flow of a pollutant in a river, or of heat in a bar, or of electrons in a wire, or almost anything which flows in a one-dimensional situation.

We need more equations which reflect the peculiarities of vehicular traffic. These equations may be balance equations (perhaps an equation for $d/dt \int_x^{x+\Delta x} q(\hat{x}, t)\, d\hat{x}$) or maybe simply some abstraction of empirical data pertaining to the physical situation at hand. If we use empirical data, then those data contain (we hope!) the essential constitution of the physical situation. Such a relation is called a *constitutive equation*. For the traffic flow problem, we have much data of the form flow rate plotted against concentration (q versus k), as shown in Figure 14.2 (see Exercise 1).

Thus we assume that $q = q(k)$. It is noteworthy that the flow of vehicles increases with increasing k for k small, while it decreases to zero as k

[1] The number k_j is the *jam concentration*, the number of cars per unit length of highway when the traffic is jammed and nothing moves.

approaches the jam concentration k_j (here just slightly more than 225 veh/mi.[1] The maximum flow rate of 1500 veh/h occurs at about 75 veh/m.

Many different forms of $q(k)$ have been fitted to the data. They range from simple forms having the above general features to others which fit the data very accurately. One of the simpler models for $q(k)$ is *Greenshields'* model given by $q = u_f k(1 - k/k_j)$, where u_f is the (empirical) free speed of the road (the speed at which a vehicle would travel if it were alone on the highway), and k_j is the jam concentration (see Exercise 2).

2. Propagation of a Disturbance

Let us consider the evolution of the traffic concentration k on a long road. First let us assume that k is a function of x and t and that q is a function of k (Figure 14.2). Then (7) can be rewritten by applying the chain rule to calculate $\partial q(k)/\partial x$:

$$\frac{\partial k}{\partial t} + \frac{dq}{dk}\frac{\partial k}{\partial x} = 0. \tag{8}$$

Now, let us consider a curve $x = x(t)$ in the xt plane on which k is constant. Such a curve is called a *characteristic* of (8) and satisfies the implicit relation,

$$k(x(t), t) = \text{constant}. \tag{9}$$

The function $x(t)$ also satisfies the differential equation obtained by differentiating (9) with respect to t,

$$\frac{\partial k}{\partial x}\frac{dx}{dt} + \frac{\partial k}{\partial t} = 0, \tag{10}$$

where once again we have used the chain rule. Along a characteristic, $k(x, t)$ satisfies both (8) and (10). Hence we must have

$$dx/dt = dq/dk \tag{11}$$

along a characteristic. Since k is constant on $x = x(t)$, so is dq/dk, and we can immediately integrate (11) with respect to t, obtaining

$$x = (dq/dk)t + x_0 \tag{12}$$

as the equation of the family of characteristics. Since k is constant on each of these curves, each curve is a straight line. Note that we assume that $dq/dk > 0$ throughout.

If we know the value of k at x_0 at $t = 0$, the value of k at each point on the line $x = (dq/dk)t + x_0$ is the same as it is at x_0. However, in terms of x and t, $x_0 = x - (dq/dk)t$. Thus

$$k(x, t) = k(x_0, 0) = k\left(x - \frac{dq}{dk}t, 0\right), \tag{13}$$

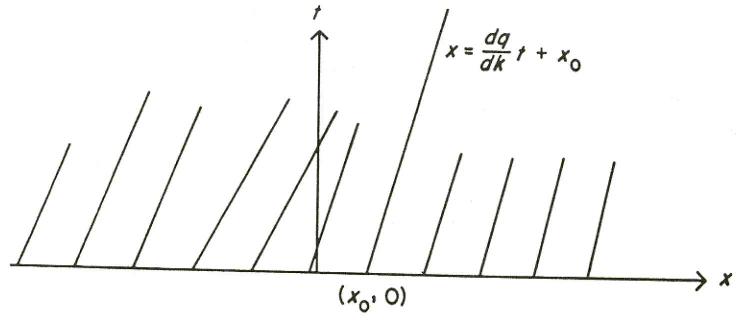

Figure 14.3. Curves of Constant k (Characteristics) (Slopes of lines are given by dq/dk for particular value of k associated with given line.)

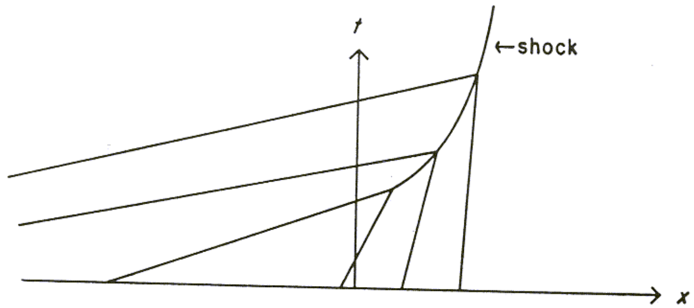

Figure 14.4. Propagation of Discontinuity

Thus if we supply the concentration of vehicles k at time $t = 0$, the initial time, the solution is determined by the relation

$$k(x, t) = k\left(x - \frac{dq}{dk}t, 0\right) \tag{14}$$

—almost. Consider the lines emanating from the neighborhood of $x = 0$ in Figure 14.3. If these lines are extended, they will cross. At a point where they cross, the equation predicts two *different* values of k. Physically, this cannot happen. Thus the partial differential equation cannot be valid everywhere along any two of the characteristic lines which cross at some point (x_1, t_1). A little thought suggests that somewhere on at least one of those characteristic lines, the solution must be discontinuous. We shall discuss the location of the discontinuity shortly. Before that, we note that the set of (x, t) where the discontinuity occurs must be such that each given line must be separated from all others which would intersect it by a discontinuity. Otherwise, two different value of k would be predicted for the point of intersection. This suggests that the discontinuity must be more than a mere point, that it must be a curve in $x - t$ space (see Figure 14.4). Such a curve of discontinuities is called a *shock*.

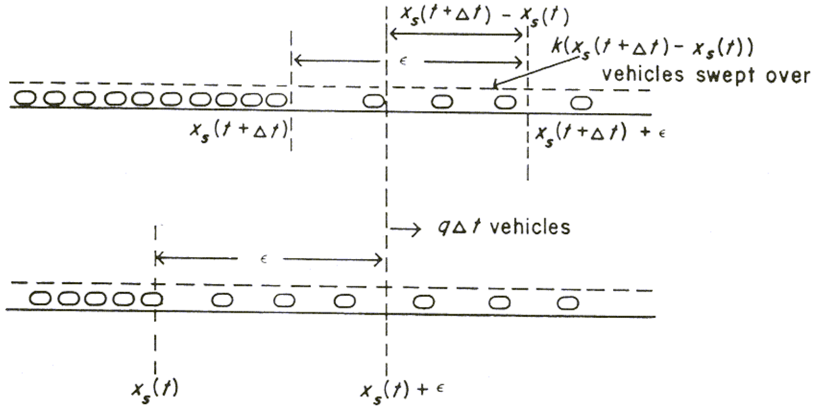

Figure 14.5. Outflow from Segment at $x_s(t) + \varepsilon$

3. Shocks

To obtain the condition valid at the shock, we must rederive the balance condition *without* the assumptions of differentiability which we made there. Consider a section of roadway which lies between $x_s - \varepsilon$ and $x_s + \varepsilon$, where x_s is the position of the shock and ε is some small positive distance. Since the shock moves in time, this section of roadway will change in time. We must account for this movement when we compute the inflow and outflow of vehicles from the segment.

First, we note that since the segment of road is small, essentially no vehicles will be found on it, and hence the rate of inflow to this segment must equal the rate of outflow from it.

Let us compute the rate of outflow from the segment of road through the end at $x_s(t) + \varepsilon$. Consider the road at two times t and $t + \Delta t$ where Δt is small (see Figure 14.5). The total number of vehicles which have flowed out of the segment between time t and time $t + \Delta t$ can be computed by considering the flow through the location $x_s(t) + \varepsilon$ during this time and subtracting those which did not make it out of the segment due to the movement of the end to position $x_s(t + \Delta t) + \varepsilon$.

Thus the number of vehicles flowing out of the segment is given by

$$q(x_s(t) + \varepsilon, t)\Delta t - k(x_s(t) + \varepsilon, t)(x_s(t + \Delta t) - x_s(t)). \qquad (15)$$

(Note that we use the flow rate and concentrations evaluated at time t. This approximation is not critical; we could have used other representative values of the time t.)

To compute the rate at which vehicles flow out, we must divide by Δt and let $\Delta t \to 0$. The rate of outflux of vehicles is then

$$q(x_s + \varepsilon, t) - k(x_s + \varepsilon, t)\frac{dx_s}{dt}(t). \qquad (16)$$

A similar calculation of the inflow at $x_s - \varepsilon$ gives

$$q(x_s - \varepsilon, t) - k(x_s - \varepsilon, t)\frac{dx_s}{dt}(t). \qquad (17)$$

Since no vehicles are created or destroyed in this segment, the difference between these two flow rates must equal the rate of accumulation of vehicles in the segment from $x_s - \varepsilon$ to $x_s + \varepsilon$. If we let $\varepsilon \to 0$, we expect no vehicles to accumulate, and thus the flow rate in (16) must equal the flow rate out of (17).

Thus we have

$$[\![q]\!] - [\![k]\!]\frac{dx_s}{dt} = 0, \qquad (18)$$

where

$$[\![f]\!] = \lim_{\varepsilon \to 0} [f(x_s + \varepsilon, t) - f(x_s - \varepsilon, t)]$$

is the *jump* in a function f across the shock.

Thus the velocity of the shock dx_s/dt is given by $[q(k_2) - q(k_1)]/(k_2 - k_1)$, where k_2 and k_1 are the concentrations ahead of and behind the shockwave. We note that $[q(k_2) - q(k_1)]/(k_2 - k_1)$ is the slope of the chord connecting the points $(k_1, q(k_1))$ and $(k_2, q(k_2))$ in the flow-concentration diagram (Figure 14.2).

If we use Greenshields' relation for $q(k)$, we find that

$$\frac{q(k_2) - q(k_1)}{k_2 - k_1} = \frac{1}{2}\left[\frac{dq}{dk}(k_1) + \frac{dq}{dk}(k_2)\right],$$

so that dx_s/dt, the velocity of the shock, is the *average* of the slopes of the characteristics which meet at the shock. Using this rule and practicing a bit, it becomes possible to sketch the characteristics and the shocks for relatively complex traffic flows.

For example, let us consider the propagation of a traffic "hump." If the hump is as shown in Figure 14.6(b), with characteristics as sketched in Figure 14.6(a), we see that a shock must form somewhere around $x = 0$ and persist, intersecting pairs of characteristics at the average of their slopes.

The situation shown in Figure 14.6 corresponds to low concentration flows with $k < k_m$, where k_m is the concentration corresponding to maximum flow. If we consider concentrations greater than k_m, the shock will propagate backward. See Exercise 3.

We should also point out that dx_s/dt is the velocity of propagation of the shock and is not related to the velocities of individual vehicles. The average speed of the traffic defined by $u = q/k$ is always positive for $0 < k < k_j$. The shock speed, on the other hand, can be positive or negative, depending on the two concentrations on either side.

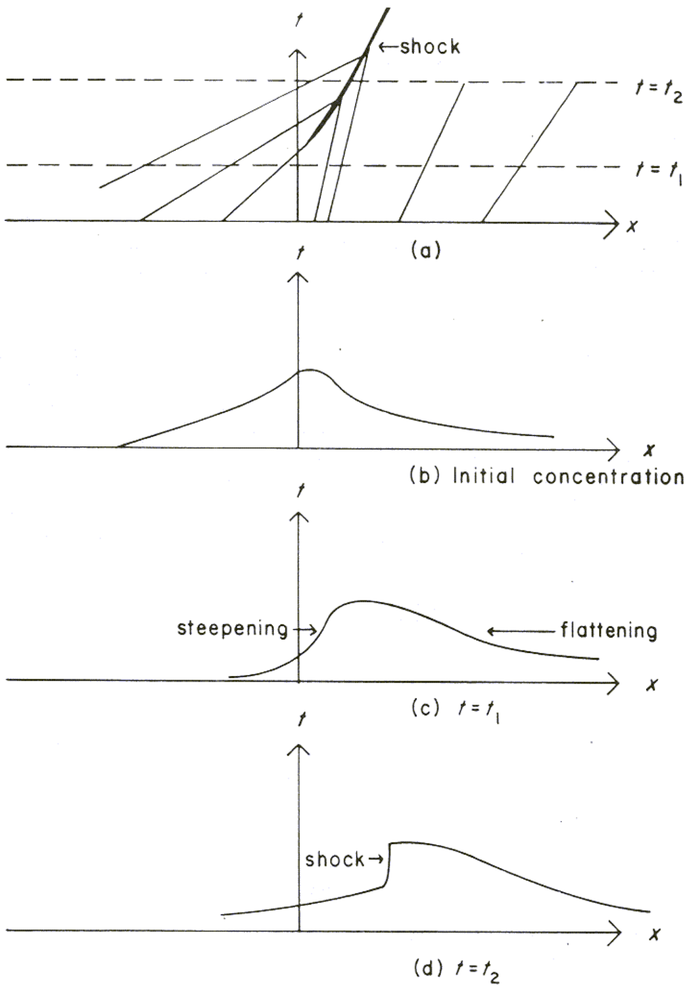

Figure 14.6. Development of Shock.

Exercises

1. Given the following data,

k (veh/mi)	$\frac{q}{k}$ (mi/h)	q (veh/h)
33	31	1023
43	26	1018
43	27	1061
48	23	1104
50	26	1300
92	12	1104
96	12	1112
98	11	1078
103	10	1030
106	10	1060
107	10	1070
110	8	880
110	9	990
114	9	1026
118	9	1062
119	9	1071
119	9	1071
121	9	1089
134	8	1072
135	8	1080
137	8	1096

use least squares to fit:

a) Greenshields' model

$$q = u_f k(1 - k/k_j);$$

b) Greenberg's model

$$q = u_m \ln (k_j/k).$$

That is, in a) choose u_f and k_j to minimize $\sum_{i=1}^{N} [q_i - u_f k_i(1 - k_i/k_j)]^2$, where (k_i, q_i) is an entry in the data table.

2. Compute the maximum flow rate for Greenshields' model. At what concentration does the maximum flow occur?

3. Consider the propagation of the traffic situation shown in Figure 14.7, where $k > k_m$. The characteristics for small t are given in Figure 14.7(b). Use your intuition about the formation of shocks to predict when a shock will form, and how it will propagate.

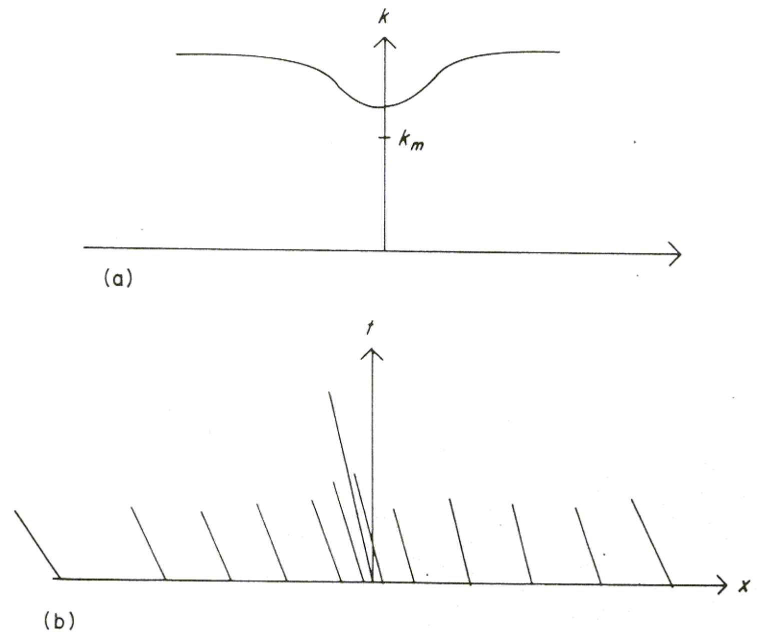

Figure 14.7. (a) Concentration at $t = 0$; (b) Characteristics Corresponding to Concentration in (a).

References

[1] F. A. Haight, *Mathematical Theories of Traffic Flow*. New York: Academic, 1963. This book is nicely mathematical, quite general, and not too hard to read. It is somewhat dated.
[2] D. R. Drew, *Traffic Flow Theory and Control*. New York: McGraw-Hill, 1968. No relation. A very general, readable book. It has many simple but important calculations pertaining to different aspects of highway design.
[3] D. C. Gazis, *Traffic Science*. New York: Wiley-Interscience, 1974. Chapter 1, written by L. C. Edie, deals with flow theories and is quite up to date.
[4] D. L. Gerlough and M. J. Huber, *Traffic Flow Theory, A Monograph*, Special Report 165, Traffic Research Board, National Research Council, Washington, D.C, 1975. Everything you always wanted to know about traffic flow theory—and more. This is an expensive paperbound monograph which synthesizes and reports, in a single document, the present state of knowledge in traffic flow theory. Not for children.

Notes for the Instructor

Objectives. This module introduces the fundamental balance idea necessary to derive the kinematic conservation equation. A discussion of constitutive equations for traffic flow is given. To illustrate the complexities of the model (and the physical situation), characteristics of first-order partial differential equations are derived and used from first principles. The modeling ideas are the main emphasis of this module.

Prerequisites. Multivariable calculus and differential equations and some perserverance should gain much understanding of modeling from this module.

Time. The module can be covered in three lectures.