Converging Concepts of Series: Learning from History

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The job of a teacher is to facilitate student learning, where "learn" is an active verb whose subject is "student." If we are to do more than just present material, we must understand the learning process, identify areas of particular difficulty, and develop teaching strategies that will help the student overcome the difficulty. Learning is not a uniformly continuous process. It happens sometimes incrementally, sometimes in more of a quantum leap. Some topics seem to build slowly on a foundation already in place, others require a major shift in thinking. This appears to be the case both for the individual student of mathematics and for the collective culture of mathematicians. The leaps required for the students are very likely to coincide with what historically was a leap for the collective culture. Using the history of such a topic to facilitate the students' leap to understanding of a topic is more than just an enjoyable diversion; it can be a sound pedagogical technique.

What I propose is a model for the development of mathematical concepts. No claim is made for its being a deep psychological model for learning, nor a major step in cognitive theory. It is a model that provides a point of view, both to understand how mathematical concepts developed historically and to plan pedagogy. The model's value will be to help identify where ideas require a major shift in understanding (which I suspect we generally categorize as "mathematical maturity"). When we are aware of where in our history the major shifts occurred, we will be more sensitive to the students' problems of trying to grasp the material. If we can understand how mathematicians accommodated a radical idea when it was first developed, we can better plan the strategy to help the student learn and accommodate it.

Model

The model I propose posits several levels of successive abstraction. The first level is people just doing things—finding the length of a side of a triangle, or circumference of a circle, or area or volume, etc.—solving a single, concrete problem, with no indication of generalizable method or theory. For example, the Old Testament passage cited for the use of 3 as a working equivalent of π [2] is:

> Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. [I Kings vii. 23]

This is specific, the numbers (arrived at by guess or measurement or approximation or whatever it takes) are given and that's the end of it. An answer is all that is sought, and if the presented value works, there is no need to explain how the answer was determined. This is the level at which the student accepts as authority the answers given in the back of the book, or tables of values, or the instructor's word.

The second level is then a method or algorithm for obtaining a solution of a specific type of problem. The method of presentation may still be in the form of a single worked example, but there is an understanding that the numbers used are just representative, that similar steps with other numbers will also work. Most of the problems in the Rhind Papyrus (c. 1650 B.C.) [6] are of this level, especially its area problems, the division of loaves or grain, and the mixture ("pesu") problems. These show that some abstracting has occurred, some common element identified that makes the problems "all the same, with only the numbers changed," rather than a collection of different problems. A solution is, in a sense, recycled.

The third level represents a higher level of abstraction, a removal away from actual physical objects; objects under discussion become more of what we would think of as defined terms rather than objects having a physical existence. The "aha" problems in the Rhind Papyrus are of this sort, for example:

> Problem 24: A quantity and its 1/7 added together become 19. What is the quantity? Assume 7. $1\frac{1}{7}$ of 7 is 8. As many times as 8 must be multiplied to give 19, so many times 7 must be multiplied to give the required number. [6, p. 66]

The quantity sought has an abstract quality different from that of applied problems, with no hint of its being a quantity *of* anything in particular. And the method of false position used to solve it represents a fairly sophisticated mathematical technique. Any convenient value is assumed for the desired quantity and is substituted into the expression, then appropriate adjustments are made to obtain the correct value.

While most of the problems may convey a sense of physical reality, that there is an actual square object, round field or pool under consideration at this third level, you begin to get concepts of idealized circles or squares. A classic example of this situation would be Greek geometry, which viewed physical observation as only the first process towards understanding reality, which in itself was a mental construct. Areas of polygons and volumes of pyramids might be used by engineers and architects, but that was not the intent of Euclid's work.

By this third level, the process is sufficiently non-obvious so that some kind of instructions are needed beyond a single numerical example. Archimedes (287–212 B.C.), approximating the area of a circle with inscribed and circumscribed polygons, must give instructions on how to proceed [9, vol. ii, pp. 50–52]. Antiphon the Sophist (c. 430 B.C.), attempting to square the circle using inscribed polygons, supplies instructions for doubling the number of sides of the polygon and how to proceed [9, vol. i, pp. 221–22].

The mention of Antiphon brings up a problem, a sign of difficulties to come. Antiphon made a leap from inscribed polygons with a finite (though large) number of sides to an infinite number of sides, from "approximate as closely as you like" to "be exactly the same," from polygonal sides *approximating* the curve to polygonal sides *coinciding* with the curve. The rejection of Antiphon's faulty reasoning [9, vol. 1, p. 221] still left the question, "How do you find areas of figures with curves for boundaries?" Special cases led to classes of figures for which areas could be determined, which led to abstract ideas of area based on the use of inscribed triangles or rectangles. Nothing worked for circles. We have the third level maturing into something different, a fourth level.

The fourth level is perhaps a maturation of the third level, a look at more sophisticated questions or problems that occur. There is a separate level of realization here, of distinctions that have to be made in the abstract objects being studied, of differences that need to be considered. The definitions and the objects under study force a reconsideration of the concepts. Archimedes used inscribed and circumscribed polygons to approximate areas of circles successfully, but recognized that they were approximations. Antiphon had mistakenly extended a similar procedure from finite to infinite number of sides, failing to realize that areas of polygons, no matter how large the number of sides, will never exactly coincide with that of a circle. A failure of the method or process used has brought up a problem. In most cases, resolution of such problems will require an increase in the level of abstraction and sophistication. In this case, a general method for finding areas of figures with curved boundaries required the development of calculus.

An analogy and pictorial representation of the model might be useful. (See Figure 1.) The analogy is a mountain path, with "altitude" signs along the way. The gentle first slope at Level 1 leads to a somewhat steeper (though straight) path up to Level 2. Level 3 looks steeper and winds a bit. When you hit Level 4, you encounter a crevasse. Frequently you are unaware of this abyss until you have almost stepped into it. To continue your ascent requires alternative action. The action may be lateral movement trying to get around it, or backtracking, or even returning to the starting point and beginning over; it may also be building a bridge across the crevasse. And what do you discover on the other side? Usually you are at another Level 1, which gives you a brief respite but which has a path leading upwards again. The mathematical version would be something like this:

Level 1: A single, concrete problem is solved, using methods improvised for this single problem. Usually there is no indication of generalizable method or theory.

Level 2: In some cases the method or algorithm used for solving a single problem can be used as or adapted as a solution for other problems of the same type. A solution is, in a sense, recycled. This recycling is the value of abstraction, which we quite frequently forget to mention to students.

Level 3: Particularly useful techniques may be applied beyond the original object of study. By isolating and abstracting key features, the method may be adapted to a larger class of

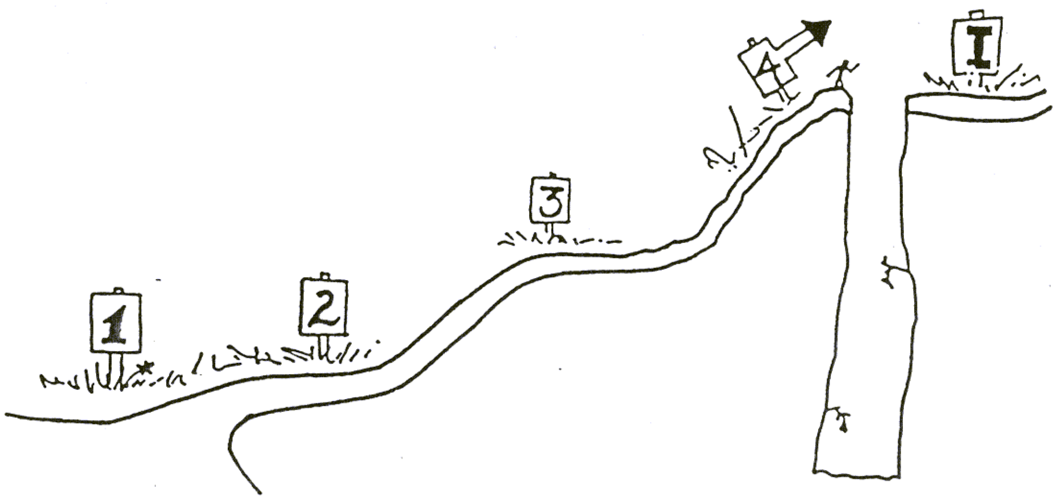


FIGURE 1

problems. There is a widening of the field of applicability. The objects under discussion become more defined terms rather than objects having physical existence; they are removed from the "intuitive" understanding of the object that gave rise to the algorithm.

Level 4: Level 3 then may lead to an exuberant and confident application to all manner of problems, until difficulties or contradictions arise. Thus Level 4 is perhaps a maturation of Level 3, a look at more sophisticated questions or problems that occur. There is a separate level of realization here, of distinctions that have to be made in the abstract objects being studied, of differences that need to be considered. A crisis may be reached where accepted methods fail or produce contradictory results—the method has been extended beyond its applicability, and some accommodation must be made. The definitions and the objects under study force a reconsideration of the concepts. This failure is illustrated by the crevasse which must be bridged.

Level 1 is a straightforward, "intuitive" stage of development, generally a relatively easy concept for students to comprehend. Getting to Level 2 requires some small mental adjustment, while Level 3 is a much larger (and to some, perhaps, impossible) cognitive shift. Level 4 "crisis" states may have required years or generations or even centuries of coping by the mathematical community, and it is not surprising that our students will have difficulty with them. Moreover, the levels are not linear, and there may be not a single path but many paths all crossing and recrossing, so you have no clear direction in which to move. There will be loops and recursion, with strategic retreats to former positions and reinstatement of formerly excluded ideas. Levels will rarely be clearly delineated; transition boundaries are very fuzzy things indeed. The same topics can be viewed as levels in the development of different concepts; we can use another metaphor and see the development as an interweaving of threads into a tapestry that makes following a single thread as difficult as sorting out this sentence.

An Example of the Model

As an example of this intricate development in both historic and pedagogical terms, I would like to use the concept of the *sum of the series*. The standard presentation of infinite series in calculus courses as taught in the United States is the following:

(1) A short introduction to infinite sequences, to prepare for sequences of partial sums;
(2) Abstract definitions of infinite series, with convergence defined in terms of limits of sequences of partial sums;
(3) Theorems and convergence tests for positive term series;
(4) Theorems and convergence tests for alternating series;
(5) Theorems and convergence tests for general series;
(6) Definition of and theorems about power series.

The instructional emphasis is on convergence and especially on tests of convergence. We spend our time finding out whether series converge or not, but little or no time finding out what the series converge *to*. And since most series that arise in applications are relatively well behaved, examples and exercises for testing convergence often comprise highly artificial and pathological cases.

That an unending collection of numbers may have a finite sum is not an easy concept to grasp. Since series are generally presented without history and separate from applications, the student must wonder not only "What are these things?" but also "Why are we doing this?"

The preoccupation with determining convergence but not the sum makes the whole process seem artificial and pointless to many students—and instructors as well. The fact is that series are greatly different from anything encountered before; and they are made harder to understand because we present solutions to Level 4 problems without first convincing the students that there is a problem. It's as if in our model we air-lifted them to the far side of the crevasse instead of leading them up the path and helping them construct a bridge. Recall that as mathematicians were first developing the concepts of series, they did not know in advance the final forms that these concepts would take. This is exactly the position our students are in: they too are developing concepts and do not know what the final form of those concepts should be. If we select topics that not only illustrate the concepts, but also trace the historical progression of ideas, we can help students make the transition to understand the Level 4 concerns we are addressing.

Let us look at a teaching strategy and a plan for sequencing topics and building around the model, looking at the several cognitive shifts which must be made. We will see that there is a shift (Level 2) in going from adding up specific numbers to finding the sum of a finite series. A shift to Level 3 can be illustrated with the move to more algebraic techniques. The first big shift in definition of "sum" will come with the move to infinite series, where the rules of arithmetic may no longer apply. By thus preparing the stage for identified cognitive shifts, the questions of "Why do we need to do things this way?" or "Why do we need to concern ourselves about this?" do not arise because the students will see the necessity for themselves.

Figurate Numbers. We may take as a starting point the Greek figurate numbers [9]. Determining the triangular numbers and their sums is a Level 1 development that students can understand immediately. Triangular numbers and their sums are tangible and concrete: they can be found by counting dots on a diagram.

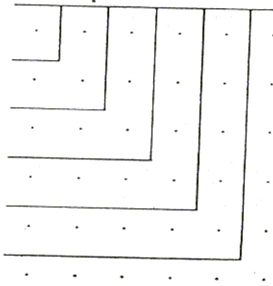
Triangular Numbers	N	Sum
·	1	1
· ·	2	3
· · ·	3	6
· · · ·	4	10
· · · · ·	5	15
· · · · · ·	6	21
· · · · · · · · ·	n	$\frac{n(n+1)}{2}$

We make a minor shift (Level 2) by looking at the nth triangular number as a sum,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

It is still familiar, though the "..." and the "n" take a little getting used to. There is no great change from simple addition, and we can put in numbers to check the formulas and reassure ourselves that all is well.

A small step takes us to square numbers:



Square Numbers	N	Sum
	1	1
	2	4
	3	9
	4	16
	5	25
	6	36

Obtaining the nth square number in a similar fashion leads to the sum of odd numbers,

$$1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2.$$

As other forms of figurate numbers were considered, the second level determination of properties common to all polygonal numbers appears, an organizing principle for triangular numbers, square numbers, pentagonal numbers, and so on. Certain sums go together, add up to the same kind of numbers which have some defining property. Nicomachus of Gerasa (c. A.D. 100) noted the pattern of numbers rather than shapes, and made a step forward in abstraction to Level 3 [13]. For Nicomachus, triangular numbers represented the sum of an arithmetic sequence with common difference 1, the square numbers a sequence with common difference of 2, pentagonal and hexagonal numbers 3 and 4 respectively. He generalized to a class of objects only abstractly similar. In this we see a move toward abstraction and away from the concrete, which can help the student make the same step. This is illustrated further by the table Nicomachus presents, which retains its touch with its geometrical roots only in the names [13, pp. 248–9]:

Triangles	1	3	6	10	15	21	28	36	45
Squares	1	4	9	16	25	36	49	64	81
Pentagonals	1	5	12	22	35	51	70	92	117
Hexagonals	1	6	15	28	45	66	91	120	153
Heptagonals	1	7	18	34	55	81	112	148	189

Nicomachus notes that "each polygonal number is the sum of the polygonal in the same place in the series with one fewer angle, plus the triangle, in the highest row, one place back in the series." This clearly represents a cognitive shift from the figures with which we began, for here we have interrelationships between pure numbers, with the defining characteristic being the generating algorithm rather than a geometrical arrangement of dots.

This shift to numbers and formulas and away from geometric figures made possible different discoveries relative to sums. Having seen that the sum of successive odd numbers was always a square, the Greeks turned to investigating cubes. When it was discovered that 2^3 is $3 + 5$, 3^3 is $7 + 9 + 11$, 4^3 is $13 + 15 + 17 + 19$, and so on, they were in a position to sum the series $1^3 + 2^3 + 3^3 + \cdots + r^3$: it was only necessary to find out how many terms of the series $1 + 3 + 5 + \cdots$ this sum of r cubes includes. The number of terms being $1 + 2 + 3 + \cdots + r$, the desired sum of the first r cubes is

$$\left[\frac{r(r+1)}{2}\right]^2$$

[9, p. 109]. This is moving into Level 3 activity. As we parallel the historical development in our presentation, we can add and hold interest in the development of formulae which are otherwise confusing and unnatural in a mere bare-bones direct presentation.

A Level 4 crisis occurred with Zeno (c. 450 B.C.), whose paradox of Achilles and the tortoise brought into question the meaning of adding an infinite number of terms. The response to Zeno was not a resolution of the problem but a backing away from any questions of the infinite or infinite processes, an avoidance of the question. It would be almost two thousand years before the Level 4 crisis of Zeno would be fully appreciated and confronted.

Algebraization of Finite Series. We are solidly on Level 3 with the continued algebraization of finite series, in the pursuit of formulas for sums of powers of successive integers. Two examples that might help ease the transition to this more abstract form come from Alhazen and one, following a slightly different developmental strand, from Jakob Bernoulli (1654–1705).

In the eleventh century, the Arab mathematician al-Haitham (c. 965–1039), known in the West as Alhazen, computed the volume of a segment of a parabola revolved about its base. This required formulas for sums of cubes and fourth powers of integers. Alhazen found sums of cubes in terms of squares and of fourth powers in terms of cubes. [7] A geometric model which depends only on equating areas of rectangles can help the modern student (and instructor) understand Alhazen's arithmetic argument and the otherwise confusing expression that results.

The rectangle has height $n+1$, and width $1+2+\cdots+n$. The area is $(n+1)(1+2+\cdots+n)$. The area can also be seen as the sum of the areas of the individual pieces, the squares and rectangles inside. The total area of squares is

$$1^2+2^2+\cdots+n^2,$$

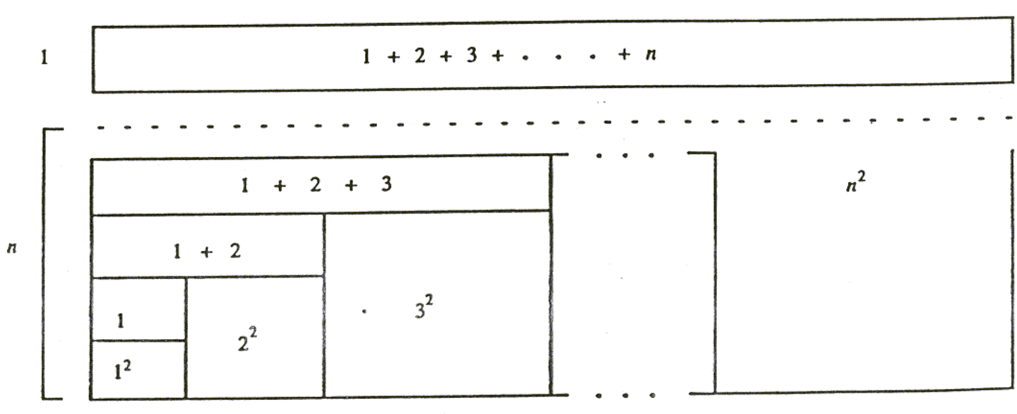


FIGURE 2
Total area equals the sum of the parts

and of the rectangles

$$1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+n).$$

We then have, in modern notation,

$$(n+1)\sum_{i=1}^{n} i = \sum_{i=1}^{n} i^2 + \sum_{p=1}^{n}\sum_{i=1}^{p} i$$

from which is found

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Now, we can use this result to get the sum of cubes:

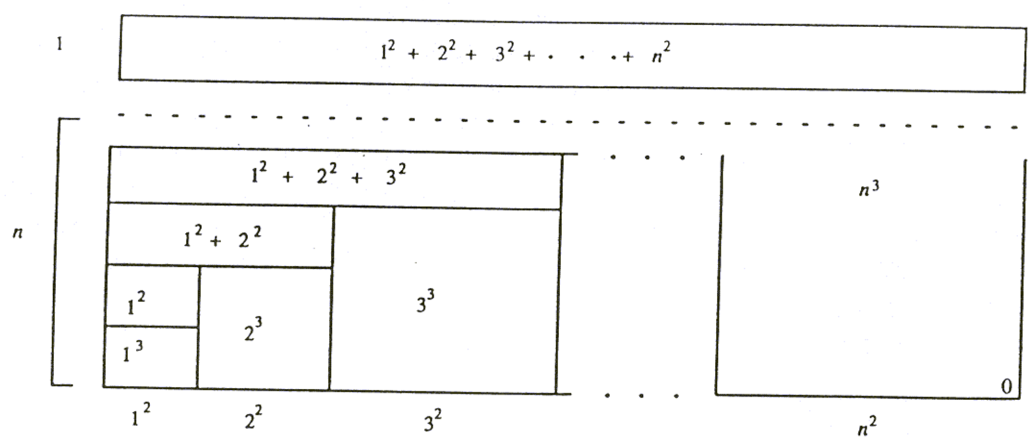


FIGURE 3

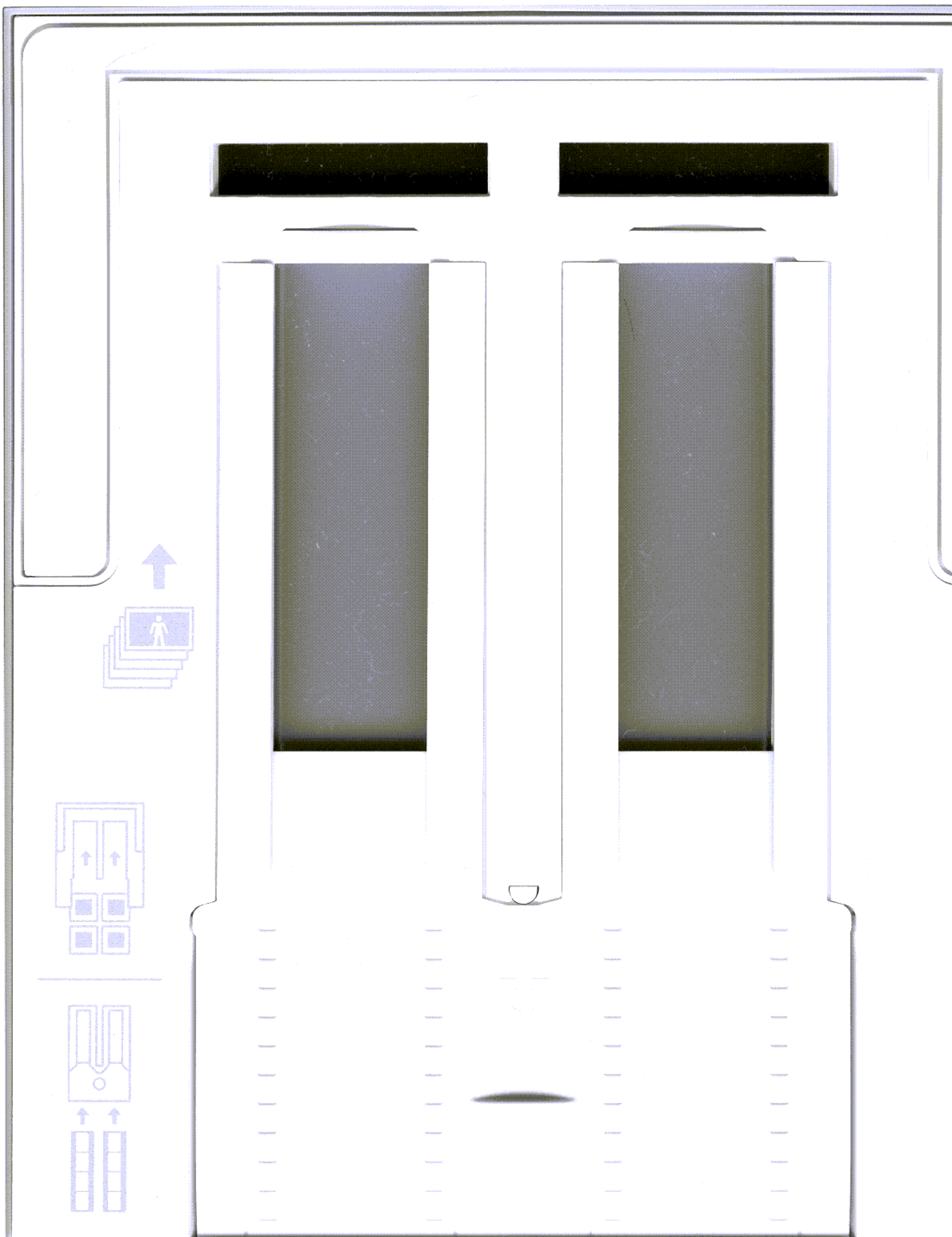
Proceeding in the same fashion as above, we get

$$(n+1)\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i^3 + \sum_{p=1}^{n}\sum_{i=1}^{p} i^2$$

from which it follows that

$$\sum_{i=1}^{n} i^3 = \frac{[n(n+1)]^2}{4}.$$

We can continue this process to give a more concrete framework on which to secure abstract concepts.



Moving from the specific cases above to a general form for the sum of kth powers:

$$(n+1)\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n}\sum_{i=1}^{p} i^k.$$

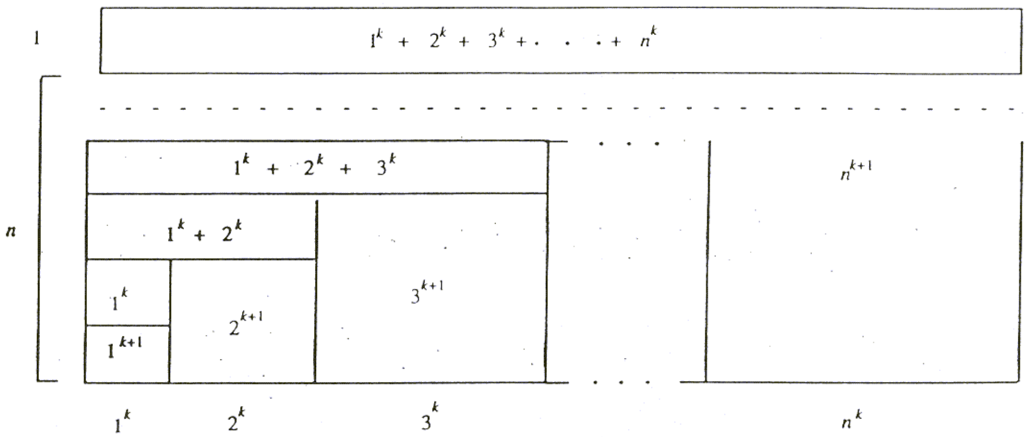


FIGURE 4

An otherwise overpowering formula can be related to familiar ideas and made not only more understandable but also more meaningful and "intuitive." Intuition requires careful development and nurturing. At least, it does if we hope to develop it in weeks rather than years or centuries!

A further historical note is appropriate, from another strand of historical development that is interwoven inextricably at this point. By the time of Pierre de Fermat (1636) the geometrical origins of figurate numbers had developed into a more algebraic form along a somewhat different path than the polygonal numbers used by Nicomachus. The first type of figurate numbers were for Fermat as the triangular numbers were for Nicomachus, as noted above. The nth triangular number was the sum of the first n integers. But in this development, the second type of figurate numbers were not square numbers but pyramidal numbers, where the nth pyramidal number was formed by summing the first n triangular numbers. In general, figurate numbers of type k were formed by summing figurate numbers of type $k - 1$. These figurate numbers form the columns of the arithmetic triangle, usually called Pascal's triangle, because of his extensive investigation of its properties.

$$\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

A general formula for the nth figurate number of type k, given by Fermat in 1636 without proof [12, pp. 229–232] was

$$\sum_{i=1}^{n} \frac{i(i+1)(i+2)\cdots(i+k-1)}{k!} = \frac{n(n+1)(n+2)\cdots(n+k)}{(k+1)!}.$$

Jakob Bernoulli (1654–1705) used a variation of the Arithmetic Triangle, making it a square in which the general form for the nth element in column k is $\frac{(n-1)(n-2)\cdots(n-k)}{k!}$, and the first k entries in column k are zeroes.

n		C_1	C_2	C_3	C_4	C_5	C_6
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	2	1	0	0	0	0
4	1	3	3	1	0	0	0
5	1	4	6	4	1	0	0
6	1	5	10	10	5	1	0
7	1	6	15	20	15	6	1
8	1	7	21	35	35	21	7

He observed [15, p. 86] that "the series of figurate numbers, supplied with the corresponding zeros, have a submultiple ratio to the series of equals." This is a chance to expose the student to the value of symbolic expressions when compared to the difficulty of working solely with words. Bernoulli is saying that the ratio of the sum of the first n terms in column k (of which the first k terms will be zero) to n times the last term, will be $1/(k+1)$. For example, from C_3:

$$\frac{0+0+0+1+4}{4+4+4+4+4} = \frac{0+0+0+1+4+10}{10+10+10+10+10} = \frac{0+0+0+1+4+10+20}{20+20+20+20+20+20+20} = \frac{1}{4}.$$

This gave him an equation relating sums of powers of integers to a known quantity. He now proceeds in a quite modern fashion to solve for the sum with the highest exponent in terms of sums of lower powers, which are already known. This is the same thing Alhazen did, but Bernoulli's development proceeds from a more algebraic perspective.

From column C_2 we have

$$\frac{0+0+1+\cdots+\frac{(n-1)(n-2)}{2}}{\frac{(n-1)(n-2)}{2}+\frac{(n-1)(n-2)}{2}+\cdots+\frac{(n-1)(n-2)}{2}} = \frac{\sum_{i=1}^{n}(i^2-3i+2)}{n(n^2-3n+2)} = \frac{1}{3}$$

$$\sum_{i=1}^{n}(i^2-3i+2) = \frac{n^3-3n^2+2n}{3}$$

$$\sum_{i=1}^{n} i^2 - \frac{3n(n+1)}{2} + 2n = \frac{n^3-3n^2+2n}{3}$$

and solving,

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

Bernoulli then used this result to derive from column C_3 that

$$\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

Except for notation, as he uses our integral sign instead of our capital sigma to denote sum, his treatment is very modern, and he gives instructions for continuing the process as far as desired.

But there is a higher level of abstraction to come. Moving from the table to the derived expressions, he establishes a pattern from the sequence of sums. Looking first at the sums of powers, he computes:

$$\begin{aligned}
\sum i &= \frac{1}{2}n^2 + \frac{1}{2}n \\
\sum i^2 &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
\sum i^3 &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
\sum i^4 &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
\sum i^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
\sum i^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\
\sum i^7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\
\sum i^8 &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \\
\sum i^9 &= \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{3}{20}n^2 \\
\sum i^{10} &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n
\end{aligned}$$

He then goes on to observe:

> Whoever will examine the series as to their regularity may be able to continue the table. Taking c to be the power of any exponent, the sum of all n^c or
>
> $$\begin{aligned}\sum i^c = {} & \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c(c-1)(c-2)}{4!}Bn^{c-3} \\ & + \frac{c(c-1)(c-2)(c-3)(c-4)}{6!}Cn^{c-5} \\ & + \frac{c(c-1)(c-2)(c-3)(c-4)(c-5)(c-6)}{8!}Dn^{c-7} + \cdots\end{aligned}$$
>
> and so on, the exponents of n continually decreasing by 2 until n or n^2 is reached. The capital letters $A, B, C, D \ldots$ denote in order the coefficients of the last terms in the expressions for $\sum i^2$, $\sum i^4$, $\sum i^6$, $\sum i^8$, ... With the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum
>
> 91,409,924,241,424,243,424,241,924,242,500

> From this it will become clear how useless was the work of Ismael Bullialdus spent on the compilation of his voluminous *Arithmetica Infinitorum* in which he did nothing more than compute with immense labor the sums of the first six powers, which is only a part of what we have accomplished in the space of a single page. [15, p. 90]

In this last paragraph, Bernoulli clearly describes the cognitive shift that has occurred, the shift from special cases to a general abstract solution, and stresses the value of the latter. This is a point teachers cannot overemphasize to their students.

Infinite Series. A Level 4 shift occurs with the move from finite sums to infinite series. So far we have reached a level where we have general forms for finite sums of powers of integers, and while there have been some major shifts—especially in treating finite sums as objects in equations—things still have close ties to arithmetic that is familiar and "normal." Adding an infinite number of terms can bring us around to Zeno again: after a thousand years there is still coping and adjusting to be done. The move to infinite series was not a single step, and there were many paths intermingled to produce the change. The one I wish to follow requires us to back up a bit first, back to the fourteenth century.

A Level 1 in infinite sums existed in medieval times. In the second quarter of the fourteenth century the natural philosophers at Merton College in Oxford, including Thomas Bradwardine (1290–1349) and Richard Swineshead (the Calculator) (fl. c. 1350), were involved with the problem of quantifying change. Their investigations into the "latitude of forms," though pursued in rhetorical fashion, presented a break from the tradition of unchanging geometric figures moving in uniform motion. One problem considered by Swineshead is relevant to our current endeavor:

> If a point moves throughout the first half of a certain time interval with a constant velocity, throughout the next quarter of the interval at double the initial velocity, throughout the following eighth at triple the initial velocity, and so on ad infinitum; then the average velocity during the whole time interval will be double the initial velocity. [7, p. 91]

In modern notation, this is equivalent to

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{n}{2^n}+\cdots=2.$$

Swineshead used a geometrical argument in his proof, to show that a sum of an infinite number of terms can have a finite value.

The fourteenth century also produces the seeds of Level 4 difficulties when, in about 1350, the natural scientist Nicole Oresme (c. 1323–1382) showed that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverged, that if the successive terms were added the whole would become infinite. He proved this by noting that $\frac{1}{3}+\frac{1}{4}$ is greater than $\frac{1}{2}$, as is the sum of the next four terms ($\frac{1}{5}$ through $\frac{1}{8}$), and the next 8 terms, and so on. We thus have an early example of a convergent infinite series with its sum, and one of a divergent series [3]. Knowledge of the latter seems not to have inhibited subsequent users of infinite series.

We can progress further onto Level 2 for infinite sums by looking at Leibniz's harmonic triangle, used in a somewhat similar fashion to the way Bernoulli used the arithmetic triangle. In 1672, Gottfried Leibniz (1646–1716) had recognized that if you take a given sequence and form a new finite sequence by taking consecutive differences of the terms of the given sequence, and then sum the terms of the new sequence, the result is the difference of the first and last terms of the original sequence. Thus, if $a_0, a_1, \ldots, a_n$ is the original sequence, and $d_i = a_i - a_{i+1}$, then

$$d_0 + d_1 + \cdots + d_{n-1} = (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{n-1} - a_n) = a_0 - a_n.$$

Leibniz realized that if the nth term of the original sequence went to zero as n increased, then the derived infinite series of differences would sum to the original first term, i.e.,

$$\sum_{i=0}^{\infty} d_i = \sum_{i=0}^{\infty} (a_i - a_{i+1}) = a_0.$$

Leibniz used this idea to solve a problem posed to him by Christiaan Huygens (1629–1695), the problem of finding the sum of the series

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \cdots + \frac{1}{n(n+1)/2} + \cdots$$

Combining his insight on sums of differences with a familiarity with the Arithmetic Triangle for figurate numbers, Leibniz formed what he called his Harmonic Triangle. Starting in the first row with the reciprocals of the integers, subsequent rows were formed from differences of consecutive terms from the preceding row.

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\
\frac{1}{2} & \frac{1}{6} & \frac{1}{12} & \frac{1}{20} & \frac{1}{30} & \frac{1}{42} & \cdots & \\
\frac{1}{3} & \frac{1}{12} & \frac{1}{30} & \frac{1}{60} & \frac{1}{105} & \cdots & & \\
\frac{1}{4} & \frac{1}{20} & \frac{1}{60} & \frac{1}{140} & \cdots & & & \\
\frac{1}{5} & \frac{1}{30} & \frac{1}{105} & \cdots & & & & \\
\cdot & \cdot & \cdot & & & & &
\end{array}$$

Just as the Arithmetic Triangle provided formulae for sums of a finite number of terms, the Harmonic Triangle provided formulae for sums of an infinite number of terms. In particular, the sum of each row is equal to the first term of the preceding row. Thus,

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1$$
$$\frac{1}{3} + \frac{1}{12} + \frac{1}{30} + \frac{1}{60} + \cdots = \frac{1}{2}$$
$$\frac{1}{4} + \frac{1}{20} + \frac{1}{60} + \frac{1}{140} + \cdots = \frac{1}{3}$$

Double the first of these gives Huygens' requested sum, which is of course 2.

What we have with Leibniz and his Harmonic Triangle is an example of Level 2, with a method applicable to a restricted class of infinite series. Though it still retains the idea of sum being related to addition as usual, yet the case of sums of differences has the look of partial sums about it. It has the added benefit of actually finding the sum of many infinite series, in a logical way that removes the mystery from the whole process.

A Level 3 move comes when series with numbers are extended to series containing variable expressions. Isaac Newton (1642–1727), working with series in *A Treatise on the Methods of Series and Fluxions* (1671, first published 1736), gives a good description of the transition that is beginning:

> Since the operations of computing in numbers and with variables are closely similar—indeed there appears to be no difference between them except in the characters by which quantities are denoted, definitely in one case, indefinitely so in the latter—I am amazed that it has occurred to no one (if you except N. Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine has the same relationship to Algebra that the doctrine in decimal numbers has to common Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root-extraction may easily be learnt from the latter's provided the reader be skilled in each, both Arithmetic and Algebra, and appreciate the correspondence between decimal numbers and algebraic terms continued to infinity ... It is the advantage of infinite variable-sequences that classes of more complicated terms (such as fractions whose denominators are complex quantities, the roots of complex quantities and the roots of affected equations) may be reduced to the class of simple ones: that is, to infinite series of fractions having simple numerators and denominators and without the all but insuperable encumbrances which beset the others. [16, pp. 33–34]

Newton is asserting that any legal operation that can be performed in arithmetic on numbers can likewise be performed in algebra on variable expressions. Just as arithmetic operations produce highly useful infinite decimal expressions, so the same operations may produce highly useful infinite series in algebra. This is a point most if not all students would agree with, and they require proof that there is anything wrong with this reasoning.

This represents a move into Level 4, however, though it is not yet recognized. The question of convergence has been blithely ignored. We see Newton working with the two series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

and

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \cdots,$$

and no distinction is drawn between them. If we substitute 1 for x in either expression, we have

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \cdots$$

in both cases. The partial sums of the expression on the right alternate between 0 (if an even number of terms are added) and 1 (if an odd number of terms are added). Trying to add together more and more terms does not lead us to any appropriate sum, certainly not $\frac{1}{2}$.

There are several ways to handle the dilemma. One is to ignore the difficulty and merely accept the result:

> In former times—before the strict foundation of infinite series—mathematicians found themselves fairly at a loss when confronted with paradoxes such as this.
>
> And even though the better mathematicians instinctively avoided arguments such as the above, the lesser brains had all the more opportunity of indulging in the boldest speculations. Thus, e.g., Guido Grandi believed that in the above erroneous train of argument which turns 0 into 1, he had obtained a mathematical proof of the possibility of creation of the world from nothing! [11, p. 133]

A somewhat less metaphysical alternative which also avoids a cognitive shift is to define the sum of a series in terms of limits of sequences of partial sums (as we do now), and dismiss as "divergent" any series that does not satisfy this convergence requirement. This is a backtrack that avoids the crevasse that has come before us.

Yet another alternative is somehow to redefine the concept of "sum" in such a way that these aberrant series can be reclaimed, building a bridge and thus making a Level 4 accommodation. Among those making the attempt to save the divergent series for analysis was Leonhard Euler (1707–1783). He attempted to redefine the meaning of "sum" in a significantly more abstract fashion, further from the then common understanding of "sum" as "to add up." In his 1760 paper on divergent series, Euler gives reasons for trying to cope with divergent series rather than dismissing them out of hand:

> Whenever in analysis we arrive at a rational or transcendental expression, we customarily convert it into a suitable series on which the subsequent calculations can more easily be performed. Therefore infinite series find a place in analysis inasmuch as they arise from the expansion of some closed expression, and accordingly in a calculation it is valid to substitute in place of the infinite series that formula from which the series came. Just as with great profit rules are usually given for converting expressions closed but awkward in form into infinite series, so likewise the rules, by whose help the closed expression, from which a proposed infinite series arises, can be investigated, are to be thought highly useful. Since this expression can always be substituted without error for the infinite series, both must have the same value. [1, p 148]

Euler summarizes Leibniz's arguments for $1 - 1 + 1 - 1 + \cdots$ being assigned the value $\frac{1}{2}$. The first is that the series is the expansion by division of $\frac{1}{1+a} = 1 - a + a^2 - a^3 + \cdots$ with a replaced by 1. The second rests on the fact that the sum of a finite number of terms is 0 for an even number of terms, 1 for an odd:

> Now if, therefore, the series is taken to infinity and (consequently) the number of terms cannot be regarded as either even or odd, it cannot be concluded that the sum is either 0 or 1, but we ought to take a certain median value which differs equally from both, namely $\frac{1}{2}$. [1, p 145]

Euler wanted to make a Level 4 accommodation, refine the definition of sum to a more abstract form, thus making the concept applicable to a wider range of series than the partial sum definition. He proposed the following definition for "sum":

> Understanding of the question is to be sought in the word "sum"; this idea, if thus conceived—namely, the sum of a series is said to be that quantity to which it is brought closer as more terms of the series are taken—has relevance only for convergent series, and we should in general give up this idea of sum for divergent series. Wherefore,

> those who thus define a sum cannot be blamed if they claim they are unable to assign a sum to a series. On the other hand, as series in analysis arise from the expansion of fractions or irrational quantities or even of transcendentals, it will in turn be permissible in calculations to substitute in place of such a series that quantity out of whose development it is produced. For this reason, if we employ this definition of sum, that is, to say the sum of a series is that quantity which generates the series, all doubts with respect to divergent series vanish and no further controversy remains on this score, inasmuch as this definition is applicable equally to convergent or divergent series. [1, p 144]

The intuitive formation of this definition of "sum" reflects an attitude still current among applied mathematicians and physicists: problems that arise naturally (i.e., from nature) do have solutions, so the assumption that things will work out eventually is justified experimentally without the need for existence sorts of proof. Assume everything is okay, and if the arrived-at solution works, you were probably right, or at least right enough. Emil Borel (1871–1956) noted this, observing that

> the older mathematicians had sufficiently good experimental evidence that the use of such series as if they were convergent led to correct results in the majority of cases when they presented themselves naturally. [4, p. 320]

This is exactly the attitude of many students—all the real problems ("real" meaning "arising in physical reality") have things working out, so why bother with the details that only show up in homework problems?

Examples for which this intuitive definition leads to problems can help meet student objections. They are instructive for discussion, even if they have already been resolved. An objection made to Euler's definition giving $\sum(-1)^n = \frac{1}{2}$, made by Jean-Charles Callet (1744–1799) in an unpublished memorandum submitted to J. L. Lagrange (1736–1813) [10], pointed out that the same series can arise from the expansion of different functions, for example,

$$\frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \cdots$$

which at $x = 1$ gives $\frac{2}{3} = \sum(-1)^n$, instead of Euler's $\frac{1}{2}$. Lagrange considered this objection and argued that Callet's example was incomplete. When the missing terms were included, the series should have been written

$$1 + 0x^1 - x^2 + x^3 + 0x^4 - x^5 + x^6 + 0x^7 - x^8 + \cdots$$

so that what was summed was

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \cdots,$$

a series whose partial sums are 1, 1, 0, 1, 1, 0, ... with an average sum of $\frac{2}{3}$. [4, pp. 319–20]

A less *ad hoc* solution to this problem was offered by G. Frobenius (1848–1917) [4, p. 319] by defining

$$\lim_{n\to\infty} \left(\sum a_n x^n\right) = \lim_{n\to\infty} \frac{S_0 + S_1 + \cdots + S_n}{n+1},$$

that is, you average the partial sums. This redefinition of "sum" is a Level 4 activity, a bridge over the crevasse.

Conditional and Absolute Convergence. We have reached the beginnings of Level 4 difficulties and the first attempts at coping. The special case of the alternating series could be explained away, but other and greater problems were arising.

Consider the following series,

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

which converges to $S = \ln 2$. We have

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \cdots$$

since the series and its convergence are unaffected by the addition of zero terms. "Adding" the original series S to this latter series term by term, we have

$$\frac{3}{2}S = 1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+0+\cdots = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$$

when the zeroes are dropped. Comparison shows this series contains the same terms, although in different order, as the original series, but its sum is $\frac{3}{2}\ln 2$. By rearranging the terms in the series, we have changed its sum. Thus, the series does not behave as a traditional sum. [5, p. 168]

Recognition that rearranging the terms of an infinite series could change its sum, first noted by A. L. Cauchy (1789–1857) in *Resumés Analytiques* (Turin, 1833) [11, p. 138], was a clear Level 4 challenge. Here is the demonstrated need for a distinction between those convergent series for which a rearrangement made no difference and those for which it did—in today's terms, between absolutely and conditionally convergent series. Riemann's Rearrangement Theorem [11, pp. 318–9], which shows that it is possible to rearrange the terms of a conditionally convergent series so that the derived series converges to any desired value, may be the final word but it ought not to be the first (or only) indication of the problem.

The above example, and others like it, could be used already in a first semester calculus class to motivate and explain the need for the distinction between conditionally and absolutely convergent series. Such examples clearly demonstrate the need for a precise definition for "sum" and the need to distinguish series for which rearrangement makes no difference and those for which it does.

Final Thoughts on Series

What I have tried to present is a way to lead a student gradually to understanding what infinite series are, based on the developmental model. The historical development of the concepts and the difficulties that forced revision and redefinition show why we have to be precise in how we define sums and how we work with them. The aim is to promote understanding while at the same time defusing the "Why are we doing this?" question. The presentation of answers without the questions is avoided, the solutions to Level 4 crevasses developed rather than presented full blown.

It would perhaps be wise to keep in mind that in the process of adapting, things that in one age are excluded from respectable mathematics may return and be reconciled. Though the accepted usage now relegates every series that does not converge (in the sense of limits of partial sums) to the category of "divergent series" and then ignores it, there is a rigorous theoretical base for summability of divergent series. Following Euler's example, the definition of sum can be made so that all previously convergent series still converge and to the same sum, but the extended definition can now apply to some divergent series. Rigorous definitions can exclude the ambiguity of the "natural" forms, and $\frac{1}{2}$ is the acceptable "sum" for the alternating series. As J. E. Littlewood wrote in the preface to Hardy's book *Divergent Series*:

> The title holds curious echoes of the past, and of Hardy's past. Abel wrote in 1828: 'Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever.' In the ensuing period of critical revision they were simply rejected. Then came a time when it was found that something after all could be done about them. This is now a matter of course, but in the early years of the century the subject, while in no way mystical or unrigorous, was regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity. [8, p. i]

Perhaps it would also be well to interject a cautionary note. It is frequently pointed out that in reading primary sources we should try to avoid judging them by "today's standard" or with the hindsight of knowledge of intervening development. But the student is coming to the material fresh, without knowing what "today's standard" is nor what the intervening developments were. Thus, if topics are developed historically the students won't—they can't—have this biased hindsight.

A major value in stressing this search for cognitive shifts through our history is that we become aware of the context of the shift and the ways of thought. We see that what was appropriate before the shift may not be appropriate after it. Minor shifts may involve slightly different methods, or slight variations in technique or ways of viewing. Major shifts involve not just totally new concepts but radically shifted bases and foundations of thought. Newton didn't just cause a change in mathematical notation; it was a change in the basis of what is a valid argument, of what is and is not "proof." The full Level 4 shift represents in its fruition more than just redefining a single term, it represents a change in our understanding of the whole concept of sum. As G. H. Hardy (1877–1947) said:

> It does not occur to a modern mathematician that a collection of mathematical symbols should have a "meaning" until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, "by X we mean Y." There are reservations to be made, ... but it is broadly true to say that mathematicians before Cauchy asked not "How shall we define $1 - 1 + 1 - \cdots$" but "What is $1 - 1 + 1 - \cdots$," and this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.
>
> It is easy now to pick out one cause which aggravated this tendency, and made it harder for the older analysts to take the modern, more "conventional," view. It generally seems that there is only one sum which it is "reasonable" to assign to a divergent series: thus all "natural" calculations with the series $[1-1+1-\cdots]$ seem to point to the conclusion that its sum should be taken to be $\frac{1}{2}$. We can devise arguments leading to a different value, but it always seems as if, when we use them, we are somehow "not playing the game." [8, p. 5]

This is the kind of adjusting our students are also trying to make, and that we are trying to help them make.

For Further Consideration

What I have presented is a teaching strategy for the concept of "sum of a series," based on my model of historical development. The time spent on each transition should increase with the level, so that a transition from level 3 to level 4 should be allowed more time than from level 1 to level 2. Other concepts that might be successfully developed according to the model in the same fashion are given below. These are intended as suggestions for possible development, not as a hard and fast formulation of how the development should be done. Any concept has many paths leading to its current form, of which one is indicated here.

1. Area:
   - Early concepts in Egypt and Babylon
   - Greek quadrature
   - Problem of arcs as boundary—Antiphon and Archimedes
   - Infinitesimals for quadrature—special techniques
   - Calculus and integration—Riemann integration, Lebesgue integration
2. The concept of "proof":
   - Single example
   - Verbose argument
   - Assumed "obviousness"
   - Appeal to authority
   - Epsilon-delta
   - Formal proofs—Russell *et al.* [Paradoxes, Theory of classes (blind alley, back up), Constructivists]
   - Gödel—we are still trying to cope
3. Convergence:
   - Sequences
   - Series of terms
   - Sequences of functions
   - Sums of functions
   - Uniform convergence
4. Number:
   - One-to-one correspondence
   - Counting numbers
   - Rationals
   - Irrationals (definitely Level 4)
   - Imaginary and complex
   - Infinitesimals—Wallis, Newton *et al.*
   - Negative numbers (another and surprisingly recent Level 4)
   - Infinitesimals again, reborn and perhaps legitimate

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