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CHAPTER 1

Setting Up First-Order Differential Equations from Word Problems

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1. Introduction

"Word problems" are sometimes troublesome; but you have learned that most noncalculus applied problems can be conquered with careful translating and attention to the kinds of units involved. A trivial illustration of this type is as follows.

EXAMPLE 1. One Sunday a man in a car leaves A at noon and arrives at B at 3:20 p.m. If he drove steadily at 55 mi/h, how far is B from A?

Solution: • distance = rate × time

$$= (55 \text{ mi/h})(3\tfrac{1}{3} \text{ h})$$

$$= (55)(\tfrac{10}{3}) \text{ mi}$$

$$\bullet = 183\tfrac{1}{3} \text{ mi.}$$

Note: In this and the other examples of this chapter, the *key* mathematical statements (equations, solutions, initial conditions, answers, etc.) are preceded by bullets (•) to stand out among the calculations. An even better way to emphasize the key statements, especially in handwritten work, would be to draw boxes around them or to use color highlighting.

Word problems involving differential equations may be more difficult than the applied problems you have dealt with heretofore. Contrast Example 1 with Example 2.

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Example 2. One Sunday a man in a car leaves A at noon and arrives at B at 3:20 p.m. He started from rest and steadily increased his speed, as indicated on his speedometer, to the extent that when he reached B he was driving at 60 mi/h. How far is B from A?

Solution: An inexperienced student might suspect that not enough information is provided. However, the steadily increasing speedometer reading means that the man's speed or velocity is a *linear function* of time, and velocity is the derivative of distance S as a function of time. So,

$$\bullet \frac{dS}{dt} = at + b \quad \text{(mi/h)}$$

and, by integration,

$$\bullet\ S = \tfrac{1}{2}at^2 + bt + c \quad \text{(mi)}.$$

If t is measured in hours, the remaining information in the problem tells us that

$$\bullet\ ① \ S(0) = 0; \quad ② \ \frac{dS}{dt}(0) = 0; \quad ③ \ \frac{dS}{dt}(3\tfrac{1}{3}) = 60; \quad ④ \ S(3\tfrac{1}{3}) = ?.$$

The first three conditions are enough to evaluate the three constants a, b, c, and the fourth will then give us the answer to the question in the problem:

$$c = 0 \quad \text{(from ①)}$$
$$b = 0 \quad \text{(from ②)}$$
$$a = 18 \quad \text{(from ③)}$$

so

$$S(3\tfrac{1}{3}) = 9t^2 = 9(\tfrac{10}{3})^2 = \bullet\ 100 \text{ mi, the distance from } A \text{ to } B.$$

Now, you may well have solved this problem differently, but all these ingredients must have been implicit in your solution. For instance, you might have realized that starting from rest would immediately give $dS/dt = at$, but you would still have been using information from the problem (condition (2)) to evaluate a constant which would otherwise have been there.

Consider another simple differential equation word problem which is commonly encountered.

Example 3. The growth rate of a population of bacteria is in direct proportion to the population. If the number of bacteria in a culture grew from 100 to 400 in 24 h, what was the population after the first 12 h?

Solution: The first sentence tells what is true at *any* instant; the second gives information on specific instants. If we denote the population by $y(t)$, the first tells us that

$$\bullet \frac{dy}{dt} = ky,$$

which has a general solution

$$\bullet\ y = A\,e^{kt}.$$

The two constants A and k may be evaluated by the information in the second sentence of the problem, that

$$\bullet\ ① \ y(0) = 100 \quad \text{and} \quad \bullet\ ② \ y(24) = 400,$$

where we have chosen hours as units for t. ① implies $y(0) = A\,e^0 = A = 100$. ② implies $y(24) = 100\,e^{24k} = 400$, which gives $k = \ln 4/24$. So

$$\bullet\ y(t) = 100\,e^{t \ln 4/24}.$$

We are asked to find $\bullet\ y(12) = 100\,e^{(12/24)\ln 4} = 200$ bacteria. (All the details of this solution are considered in Exercise 1.)

Note: We chose the units for t to be *hours*. The problem could be solved equally well with the *day* as the unit for t (then 24 h gives $t = 1$ and 12 h gives $t = 1/2$). As long as you are *consistent* throughout the problem, the choice is yours. In Exercise 1 you can confirm that you will get the same final answer of population = 200 by measuring time in days instead of hours, although you will encounter different values for some of the constants en route.

You can already see that word problems involving differential equations tend to require even more translating than Example 1 and that you cannot plug all the numbers at once into one equation. Let us summarize the pattern we have followed in Examples 2 and 3.

For a situation involving a quantity y which is going to depend on time t, we set up an equation that gives the relation between y', y, and t at *any particular moment* t. From this equation, integration gives a new equation, involving y and t, not y'. This new equation contains constants of integration and is still true for *any* specific t. *Now* information given in the problem that is true only at *specific* times (e.g., "the train's brakes failed at 11:22 a.m.") is used to evaluate the constants of integration and any other parameters (linearity, proportionality, etc.). Then at last we have a function $y(t)$ which can be evaluated immediately for any additional values of t. We shall expand upon this pattern in the next section.

2. Guidelines

Given the infinite number of guises in which word problems occur, we cannot learn a foolproof method of setting up and solving all of them, but we can list crucial areas to which you must pay careful attention.

Keep in mind that you are trying to find a *solution curve* for a differential equation. The idea behind differential equations is that if you know the derivative at each point on a curve *and* where the curve begins, then you can reconstruct this solution curve.

1) *Translation.* Many common words occur which indicate "derivative", such as "rate," "growth" (in biology and population studies), "decay" (in radioactivity), and "marginal" (in economics). Words like "change," "varies," "increase," and "decrease" are signals to look carefully at *what* is changing—again, a derivative may be called for. (See Example 5 in the next section.)

Ask yourself whether any principles or physical laws govern the problem under consideration. Are you expected to apply a known law, or must you derive what is appropriate for the problem? Answers to these questions are important guides on how to proceed. It can be particularly difficult to see how to begin the solution when you first read a problem in some field with which you are not familiar. We pay special attention to this in the examples that follow.

Many problems fall into the following pattern:

$$\begin{matrix}\text{net rate} \\ \text{of change}\end{matrix} = \begin{matrix}\text{rate of} \\ \text{input}\end{matrix} - \begin{matrix}\text{rate of} \\ \text{outgo.}\end{matrix}$$

If you can recognize this pattern when it occurs (and if you keep the physical units straight), the differential equation will probably fall out in your lap.

2) *The differential equation.* The differential equation is an *instantaneous* statement, which must be valid at *any time.* This is the central part of the mathematical problem. If you have seen the key words indicating derivatives, you want to find the relation between y', y, and t. Try to focus on the overall relations in *words* first, such as "rate = input − outgo." Write *this* down, and then make sure you fill in all the pieces listed.

3) *Units.* Once you have identified which terms go into the differential equation, make sure each term has the same physical units (e.g., gal/min or furlongs/fortnight), for real life (and textbook) exercises do not usually happen that way by themselves. Attention to the physical units can often help you out in completing the differential equation itself (as you shall see in Example 5 below).

4) *Given conditions.* These are the bits of information about what happens to the system *at a specific time.* They are held *out* of the differential equation. They are used to evaluate all the constants hanging around after the differential equation has been solved. These are the constants of proportionality or whatever in the original differential equation plus the constants of integration which occur in its solution.

These given conditions should be written *with* the differential equation in order to give completely and succinctly the mathematical statement of the problem. (In Example 2 and 3 they were delayed to illustrate their necessity.)

5) *Conceptual framework.* As noted, a bullet (•) signals the key mathematical statements in all our examples, to help you pick up the main steps that constitute a solution. In a typical problem, the key steps are completed as you successively obtain the following:

a verbal equation conceptualizing the situation in words;
a statement of any principle or physical law involved;
the differential equation;
the given conditions, initial or otherwise;
solution to the differential equation;
solution with the constants evaluated;
answer to the question of the problem.

You need to look for all of these key steps. We call this collection the "framework." The goal at each step in the problem is the completion of the next framework piece.

The fact that the computational pieces of these applied problems tend to get lengthier and more complicated as you delve more deeply into the world of differential equations (DE's) is what causes unorganized souls to lose the war with word problems. You can easily spend so many pages on a calculation that when you finally get it simplified, you have forgotten that it was only a little piece of the initial question. This is particularly true for higher order nonhomogeneous linear DE's, or for systems of DE's, or for partial DE's. So *now* is the time to learn to be organized.

You want to know at a glance *where* you are in the problem. It gets increasingly risky to do pieces in your head, but if you must, a written framework is important. If you expect others to be able to read your work, a clearly written framework is *essential.*

One good way to start a word problem is to *write down* everything you know about it. Box in, bullet, or otherwise note the key statements that contribute to the framework. Then go after the other parts of the framework list.

We have barely mentioned the solution of the differential equations in these guidelines because that is not the goal of this chapter. We are concentrating on translating the applied problem into a form whereby only routine calculations remain. Our techniques may seem overdone for simpler problems, but the aim is to help you conquer the tough ones.

This is about all that can be said in general on setting up word problems requiring first-order differential equations. Keep in mind the five guidelines—

translate
make an instantaneous statement
match physical units
state given conditions
write a clear framework

—as you work toward a solution. The only way to get more of a feeling for the process is to try working problems.

You should be able to dig into the exercises right now. Some examples are given in the following section which may or may not be helpful at the places where you get stuck. We can only show *one* possible train of thought.

3. Examples

You will get the most out of the examples if you will *try* them before reading the solutions. The aim is for *you* to be able to do word problems, not just to read one way someone else can do them. You may well create a correct solution which does not look like ours (as in Exercise 1). The check is whether you get the same *answer* to the question. If you do not, try to see what you forgot to include or where you went wrong. If you have no luck, *ask* someone.

EXAMPLE 4. An indoor thermometer, reading 60°, is placed outdoors. In 10 min it reads 70°; in another 10 min it reads 76°. Using no calculations, guess the outdoor temperature. Then calculate the proper answer, assuming Newton's Law of Cooling.

Newton's Law of Cooling—or Warming—says that when an object at any temperature T is placed in a surrounding medium at constant temperature m, then T changes at a rate which is proportional to the difference of T from the temperature of the surrounding medium. The assumption for this mathematical model is that the medium is large enough so that m is essentially not disturbed by the introduction of the warmer or colder object. Experiments have shown that this is a good approximation.

Solution: Obviously, the first order of business for the word problem is to find out what Newton's Law means; that has been provided. So we have two paragraphs from which to construct our solution.

You should zero in on the key words "changes at a rate" in the second paragraph. That sentence says that dT/dt is proportional to $T - m$, giving • $dT/dt = k(T - m)$. Three specific conditions are provided:

$$\bullet\ T(0) = 60,\ T(10) = 70,\ T(20) = 76,$$

using minutes for t and degrees for T. The solution to the differential equation is • $T = A\,e^{kt} + m$, and the three given conditions will be enough to evaluate the three constants A, k, m. (See Exercise 2.)

Now what was the question the problem asked? Simply, • what is m? As we have noted, sufficient information is given. We can also note that, at least for this problem, you need not bother finding A or k (unless you need them on the way to m). You should note, however, that k must be negative in order for T to approach a constant m as t increases indefinitely, so it would be smart to confirm that this is indeed the case.

EXAMPLE 5. A man eats a diet of 2500 cal/day; 1200 of them go to basal metabolism (i.e., get used up automatically). He spends approximately 16 cal/kg/day times his body weight (in kilograms) in weight-proportional exercise. Assume that the storage of calories as fat is 100% efficient and that 1 kg fat contains 10,000 cal. Find how his weight varies with time.

This particular application is probably less familiar to you than the earlier examples. Therefore, it is an excellent one to try by yourself. Cover up the rest of this example and try working it as programmed learning. Whenever you get stuck, move your covering paper down till you are unstuck and see if you can go on from there.

Solution: None of our super key "derivative" words appear, but we can focus on the final question, which tells us that • we want to get weight (call it w) as a function of time. If we consider w as a *continuous* function of t, we can seek a differential equation involving dw/dt.

Time comes into the problem only as "per day," so you can focus on one day and try for conceptual statements such as

each day, change in weight = input − outgo;
input will be net weight intake, above and beyond basal metabolism;
outgo will be loss due to weight-proportional exercise (WPE).

Since we are aiming for a derivative, the above can be combined in a better conceptual statement:

• change in weight/day = net intake/day − WPE/day.

This has fine form for a framework statement; we can start filling in the pieces.

daily net intake = 2500 cal/day eaten − 1200 cal/day used in basal metabolism
= 1300 cal/day.

daily net outgo = 16 (cal/kg)/day × w kg in WPE
= $16w$ cal/day.

$$\text{change in weight/day} = \frac{\Delta w}{\Delta t}\ \text{kg/day}$$

$$= \frac{dw}{dt}\ \text{kg/day, in the limit as } \Delta t \to 0$$

(which is what we need for an *instantaneous* statement about a continuously changing function $w(t)$).

As you may have noticed, some of these quantities are given in terms of *energy* (calories) and others in terms of *weight* (kilograms). What are you going to do about the fact that the units on the left of the framework statement (kg/day) do not match, those arising on the right (cal/day)? That is where the last sentence of information comes in, giving cal/kg. We can use

$$\text{kg/day} = \frac{\text{net cal/day}}{10{,}000 \text{ cal/kg}}.$$

So, filling in all the pieces gives

$$\bullet\ \frac{dw}{dt} = \frac{(2500 - 1200) - 16w}{10{,}000} \tag{1}$$

which in physical units checks out as follows:

$$\frac{\text{kg}}{\text{day}} = \frac{\text{cal/day} - ((\text{cal/kg})/\text{day})(\text{kg})}{\text{cal/kg}}.$$

How many constants are you going to have in your solution? One, from integration. So how many given conditions would you need to give a numerical answer for the man's weight on a given day? One, e.g., that at • $t = 0$, $w = w_0$, giving his weight at the beginning.

The problem of Example 5 is now completely set up for the routine calculations to take over, but we shall follow through on the solution to this problem in order to pursue some questions of interpretation and their consequences. The differential equation (1) can easily be solved by separation of variables—go ahead:

$$\frac{dw}{1300 - 16w} = \frac{dt}{10{,}000}$$

$$-\frac{1}{16}\ln|1300 - 16w| = \frac{t}{10{,}000} + C.$$

(Physically, we must have intake ≥ outgo, so we can drop absolute value sign.)[1]

$$\begin{aligned} 1300 - 16w &= e^{-16((t/10{,}000)+c)} \\ &= Q\,e^{-16t/10{,}000} \\ &= (1300 - 16w_0)\,e^{-16t/10{,}000}. \end{aligned}$$

(The given condition is an initial one, with $t = 0$, which makes the constant especially easy to evaluate.)

Solving for w,

$$\bullet\ w = \frac{1300}{16} - \left(\frac{1300 - 16w_0}{16}\right)e^{-16t/10{,}000}\ \text{kg}. \tag{2}$$

Thus we have answered the question posed by the problem, but consider one likely additional question: "Does the man reach an equilibrium weight?"

[1] Alternatively, the reader may keep the absolute value sign and show that $|1300 - 16w| = |1300 - 16w_0|\exp(-16t/10{,}000)$. From this, one concludes that $1300 - 16w$ has the sign of $1300 - 16w_0$ since the exponential factor is positive.

This question can be answered from (2) by noting that as $t \to \infty$, the right-hand term of this expression for w goes to 0, so $w \to 1300/16$ kg. However, we can also answer this last question directly from the differential equation (1). At an equilibrium, w does not change, so $dw/dt = 0$. This gives very directly that

$$w_{\text{equil}} = \frac{1300}{16}\ \text{kg}.$$

So, if the equilibrium were all we needed to know, we would not have had to solve the differential equation! We would have been finished one line after (1).

Example 6. What rate of interest payable annually is equivalent to 6% continuously compounded?

This problem can be solved *very* quickly using the differential equation idea, as shown below. However, some people do not find this line of thought very natural in the context of bank interest. The problem can also be attacked directly and more traditionally as an extension of simple interest; this method is outlined in Exercise 3.

Solution: • Let $S(t)$ be saving at time t. $S(t)$ includes the interest continuously compounded. At • $t = 0$, let $S = S_0$, which simply labels the initial amount of money, the principal.

Now then, what is the question? Behind the scenes, the problem is asking how much money will have been gained in one year. If at • $t = 1$ we let $S = S_1$, then the

$$\bullet\ \text{equivalent annual rate} = \frac{\text{money earned}}{\text{original amount}} = \frac{S_1 - S_0}{S_0}.$$

"Rate of interest" means dS/dt, for the instantaneous change per unit time in savings S is due only to the calculation of the interest at that instant. Hence, because of the continuous compounding, • $dS/dt = 0.06S$ at any instant. [Another way of stating this differential equation is as follows: dS/dt = rate of change for total amount in savings, so the rate per dollar in the account $= 0.06 = (dS/dt)/S$.] The differential equation has a general solution • $S = A\,e^{0.06t}$. (You can verify this in Exercise 1, letting $y = S$ and $k = 0.06$.)

We need a condition to evaluate A, which we provided by setting $S = S_0$ at $t = 0$. Plugging in this initial condition gives • $S = S_0\,e^{0.06t}$. Recall that we are looking for the annual rate $(S_1 - S_0)/S_0$. First find $S_1 = S_0\,e^{0.06} = 1.0618S_0$. Then

$$\frac{S_1 - S_0}{S_0} = \frac{(1.0618 - 1)S_0}{S_0} = 0.0618$$

or • 6.18% annual rate. Notice that, for this particular question, we do not need a value for S_0, one of the constants. Mathematically, that is to say that the answer requested is *independent* of S_0.

Example 7. Human skeletal fragments showing ancient Neanderthal characteristics are found in a Palestinian cave and are brought to a laboratory for carbon dating. Analysis shows that the proportion of C^{14} to C^{12} is only 6.24% of the value in living tissue. How long ago did this person live?

(Carbon dating: The carbon in living matter contains a minute proportion of the radioactive isotope C^{14}. This radiocarbon arises from cosmic ray bombardment in the upper atmosphere and enters living systems by exchange processes, reaching an equilibrium concentration in these organisms. This means that in living matter, the amount of C^{14} is in constant ratio to the amount of the stable isotope C^{12}. After the death of an organism, exchange stops, and the radiocarbon decreases at the rate of one part in 8000 per yr.)

Solution: Carbon dating enables calculation of the moment when an organism *died*; therefore, our question actually means "how long ago did this person die?" If we let • t = year *after death*, and • $y(t)$ = ratio C^{14}/C^{12} (say in $mg\ C^{14}/mg\ C^{12}$), then the last sentence of the carbon dating paragraph yields our differential equation (identified by the key word "rate"):

$$\bullet\ \frac{dy}{dt} = -\frac{1}{8000}\text{yr} \quad (\textit{decreases})$$

in $(mg\ C^{14}/mg\ C^{12})$/yr. (Another way of stating this property of radioactive disintegration is that "the rate of disintegration of a radioactive substance is proportional at any instant to the amount of the substance present." The constant of proportionality for C^{14} was given by the "one part in 8000/yr.")

Our solution will have but one constant, from integration, so one given condition will suffice for evaluation. This can be provided by noting that at the time of death of the organism, when • $t = 0$, then $y = y_0$, the proportion of C^{14} in living matter.

The general solution to the differential equation is

$$\bullet\ y = k\,e^{-t/8000}.$$

Two steps remain; to evaluate k and to answer the question.

The initial condition tells us that $k = y_0$, so we have

$$\bullet\ y = y_0\,e^{-t/8000}.$$

The question asks us to • find t when $y = 0.0624y_0$:

$$0.0624y_0 = y_0\,e^{-t/8000}$$

$$t = -8000 \ln 0.0624 \approx \bullet\ 22{,}400 \text{ yr ago}$$ (that is, the number of years before the analysis that death occurred).

Note: Recently, the practice of carbon dating has been questioned—dates between 2500 and 10,000 years ago have been in discrepancy with other dating methods. In 1966 Minze Stuiver of the Yale Laboratory and Hans E. Suess of the University of California at San Diego reported establishment of the nature of errors in carbon dating during this period. Evidently, cosmic ray activity decreased at the time, with the peak discrepancies occurring about 6000 years ago. The researchers' conclusions were the result of carbon dating of Bristlecone pine wood, which also provided accurate tree-ring dating. They suggested an apparently successful formula for correcting the carbon-dating calculation between 2300 and 6000 years ago[2]:

$$\text{true time} = C^{14}\ \text{yr} \times 1.4 - 900.$$

Example 8. A right circular cylinder of radius 10 ft and height 20 ft is filled with water. A small circular hole in the bottom is of 1-in diameter. How long will it take for the tank to empty?

We need a physical assumption about the velocity with which the water leaves the hole. Even if you feel far removed from physics, consider the following. It certainly is reasonable to assume this velocity will depend on $h(t)$, the height of the water remaining in the tank at time t. After all, the water will flow faster when the tank is full than when it is nearly empty (the greater depth of water exerts more pressure to push water out of the hole). Furthermore, if one assumes no energy loss, then the potential energy lost at the top when a small amount of water has left the tank must equal the kinetic energy of an equal amount of water leaving the bottom of the tank through the hole. That is,

$$mgh = \tfrac{1}{2}mv^2,$$

at any instant, so

$$\bullet\ v = \sqrt{2gh},$$

where g is the acceleration due to gravity, which is exactly the relation cited in physics as *Torricelli's Law*. Physically, this model may be an oversimplification for the situation in question, but at the very least we can agree that the dependence on height seems reasonable. Further physical argument might possibly produce a better formula for velocity, but it would not otherwise change the *mathematical* analysis, which proceeds as follows.

[2] An interesting, readable, and detailed account of dating procedures is contained in Louis Brennan's *American Dawn, A New Model of American Prehistory*; (New York: Macmillan, 1970, ch. 3). Stuiver and Suess originally reported their work in the professional journal *Radiocarbon*; the results are summarized in *American Antiquity*, July, 1966. For our purposes, we merely take note of the reassurance that for dating more than 10,000 years or less than 2500, the carbon dating model of this Example is found to be quite accurate (that is, within 200 years).

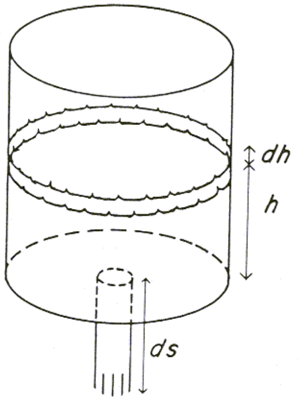

Figure 1.1

Solution: The situation says to look at *volumes*. The volume of water in the tank decreases by the volume which escapes through the hole. Letting A be the horizontal area of the tank and B be the horizontal area of the hole,

$$-A\,dh = +B\,ds$$

(volume in tank decreases) (volume leaving tank increases)

in any interval dt. Now what is our question? • When (i.e., at what t) does $h = 0$? Therefore, we want to find $h(t)$. So far we have but one relation, which can be rewritten $dh = -(B/A)\,ds$. We can calculate A and B, but what can we do with ds?

The other information we have is the *velocity* creating ds, so we can make the following substitution:

$$ds = \left(\frac{ds}{dt}\right) dt = v\,dt$$

which gives

$$\bullet\ dh = -\frac{B}{A} v\,dt.$$

We can calculate the following:

$$A = \pi(10)^2 \text{ ft}^2$$

$$B = \pi(1/24)^2 \text{ ft}^2$$

$$v = \sqrt{2gh} = \sqrt{2(32)h} = 8h^{1/2} \text{ ft/s}.$$

Substitution and separation of variables gives

$$\bullet\ h^{-1/2}\,dh = \frac{-8(1/24)^2}{(10)^2}\,dt$$

so

$$2h^{1/2} = \frac{-8(1/24)^2}{(10)^2} t + K.$$

When • $t = 0$, $h = 20$, so $K = 2\sqrt{20}$. We want t when $h = 0$, so

$$t = \frac{(100)}{8(1/24)^2}(2\sqrt{20}) \approx \bullet\ 18 \textit{ hours} \text{ (or 64,800 s)}.$$

Exercises

(The most challenging exercises are denoted by a dagger.)

1. Fill in the following details of Example 3:
   a) Solve $dy/dt = ky$ (by separation of variables) to get $y = A\,e^{kt}$.
   b) Show that $y(24) = 400$ implies that $k = (\ln 4)/24$.
   c) Show that $y(12) = 200$.
   d) Show that you get the same answer of population = 200 after 12 h if you let the unit of time be one *day*.

2. Fill in the following details of Example 4:
   a) Solve $dT/dt = k(T - m)$ to get $T = A\,e^{kt} + m$.
   b) Using the three given conditions, find m.

3. Try another method to handle Example 6. Again, let $S(t)$ be savings as a function of time t, *measured in years*.
   a) Then *simple* interest, payable *annually*, at rate r would mean $S_1 = S_0 + rS_0$. Show that after n years $S_n = (1 + r)^n S_0$.
   b) Interest compounded *quarterly* at an annual rate r would mean after one quarter that $S(1/4) = S_0 + (r/4)S_0$. Show that after n quarters, $S(n/4) = (1 + r/4)^n S_0$.
   c) Interest compounded *daily* at an annual rate r would mean after one day that $S(1/365) = S_0 + (r/365)S_0$. Show that after n days, $S(n/365) = (1 + r/365)^n S_0$.
   d) Interest *continuously compounded* at a rate r is computed by $\lim_{m\to\infty} S(n/m)$. Show that this implies, at an instant when $t = n/m$, that $S(n/m) = (e^r)^{n/m} S_0$, i.e., that $S(t) = S_0 e^{rt}$.

4. What is the half-life of C^{14}? (See Example 7. Half-life is the time required for half the amount of the radioactive isotope to disintegrate.)

5. At Cro Magnon, France, human skeletal remains were discovered in 1868 in a cave where a railway was being dug. Philip van Doren Stern, in a book entitled *Prehistoric Europe, from Stone Age Man to the Early Greeks* (New York: W. W. Norton, 1969), asserts that the best estimates of the age of these remains range from 30,000 to 20,000 B.C. What range of laboratory C^{14} to C^{12} ratios would be represented by that range of dates? (See Example 7.)

6. By Newton's law, the rate of cooling of some body in air is proportional to the difference between the temperature of the body and the temperature of the air. If the temperature of the air is 20°C and boiling water cools in 20 min to 60°C, how long will it take for the water to drop in temperature to 30°C?

7. A fussy coffee brewer wants his water at 185°F, but he almost always forgets and lets it boil. Having broken his thermometer, he asks you to calculate how long he should wait for it to cool from 212° to 185°. Can you solve his problem? If you answer "yes," do so. If "no," explain why.

8. Water at temperature 100°C cools in 3 min to 90°C, in a chamber at 60°C. Temperature changes most rapidly when the temperature difference between the water and the room is the greatest. Experiments show the rate of change is linearly proportional to this difference.
   a) Find the water temperature after 6 min.
   b) When is the temperature 75°C? 61°C?

†9. One ounce of water at 90°C is set afloat in a plastic cup in a photographer's chemical solution of exactly 100 oz at 10°C. This is an effort to warm the solution without diluting it.
   a) Express the temperature of the solution as a function of time.
   b) Reconcile your model with the temperature model used in Example 4.

10. A spherical raindrop evaporates at a rate proportional to its surface area. Find a formula for its volume V as a function of time.

11. A 100-gal tank is filled with water and 20 lb of salt. Fresh water is pumped in at a rate of 2 gal/min. The mixture is continuously stirred, and overflows to keep the tank at the 100-gal level. How much has the concentration of salt been diluted after one hour?

12. Water pollution can be diminished by treatment of raw sewage before it reaches the water supply. A common method is to use an activated sludge aeration tank containing a concentration c (which varies with time) of pollutant. Raw sewage containing a greater concentration c_1 of the pollutant enters the tank, bacteria digest some of the sewage, and the resultant cleaner mixture is dumped into a water body. The concentration of pollutant in the discharge must not exceed a certain safe level, say $0.30c_1$, so the problem is to find the time when that level is reached. In practice, at that t the raw sewage can be diverted to another tank, while this one is aerated to reduce c to c_0, the reasonable minimal level. Assume that the tank receives input at a rate of r_1 gal/min and the effluent leaves at a rate of r_2 gal/min. At $t = 0$ the tank holds V_0 gal of sewage containing z_0 lb of pollutant. Set up the problem for a mathematician to solve in order to find the time at which this tank should be bypassed.

13. If a savings account, with interest continuously compounded, doubles in 16 yr, what is the interest rate?

†14. A college education fund is begun with $\$P$ invested to grow at a rate r continuously compounded. In addition, new capital is added every year on the anniversary of the opening of the account, at a rate of $\$A$/yr. Find the accumulated amount after t yr.

15. A tank is filled with 10 gal of brine in which 5 lb of salt is dissolved. Brine having 2 lb of salt per gallon enters the tank at 3 gal/min, and the well-stirred mixture leaves at the same rate.
    a) What is the concentration of salt in the water leaving the tank after 8 min?
    b) How much salt is in the tank after a long time?

16. Neutrons in an atomic pile increase at a rate proportional to the number of neutrons present at any instant (due to nuclear fission). If N_0 neutrons are initially present and N_1 and N_2 neutrons are present at times T_1 and T_2, respectively, show that

$$\left(\frac{N_2}{N_0}\right)^{T_1} = \left(\frac{N_1}{N_0}\right)^{T_2}.$$

17. Water containing 2 oz of pollutant/gal flows through a treatment tank at a rate of 500 gal/min. In the tank, the treatment removes 2% of the pollutant per minute, and the water is thoroughly stirred. The tank holds 10,000 gal of water. On the day the treatment plant opens, the tank is filled with pure water. Find the function which gives the concentration of pollutant in the outflow.

†18. During what time t will the water flow out of an opening 0.5 cm² at the bottom of a conic funnel 10 cm high, with the vertex angle $\theta = 60°$?

19. At time $t = 0$, two tanks each contain 100 gallons of brine, the concentration of which then is one half pound of salt per gallon. Pure water is piped into the first tank at 2 gal/min, and the mixture, kept uniform by stirring, is piped into the second tank at 2 gal/min. The mixture in the second tank, again kept uniform by stirring, is piped away at 1 gal/min. How much salt is in the water leaving the second tank at any time $t > 0$?

20. Modeling glucose concentration in the body after glucose infusion: Infusion is the process of admitting a substance into the veins at a steady rate [this is what happens during intravenous feeding from a hanging bottle by a hospital bed]. As glucose is admitted, there is a drop in the concentration of free glucose (brought about mainly by its combination with phosphorous); the concentration will decrease at a rate proportional to the amount of glucose. Denote by G the concentration of glucose, by A the amount of glucose admitted (in mg/min), and by B the volume of liquid in the body (in the blood vessels). Find whether and how the glucose concentration reaches an equilibrium level.

†21. A criticism of the model of Exercise 20 is that it assumes a constant volume of liquid in the body. However, since the human body contains about 8 pt of blood, infusion of a pint of glucose solution would change this volume significantly. How would you change this model to account for variable volume? I.e., how would you change the differential equation? Will this affect your answer about an equilibrium level? How? What are the limitations of *this* model? (Aside from the fact you may have a differential equation which is hideous to solve or analyze, what criticisms or limitations do you see physically to the variable volume idea?) What sort of questions might you ask of a doctor or a biologist in order to work further on this problem?

22. A chemical A in a solution breaks down to form chemical B at a rate proportional to the concentration of uncoverted A. Half of A is converted in 20 min. Express the concentration y of B as a function of time and plot it.

23. A limnology class is presented with a laboratory exercise concerning continuous culture of algae in a chemostat. The apparatus consists of a culture vessel (with a constant level overflow tube to keep the volume at 8 liters) into which a fresh culture medium is continuously fed by a constant metered gas flow. A page of instructions is handed to the students. It contains physical and numerical data for the experiment, and concludes with the following paragraph:

"With the pumping of a fresh culture medium into the culture chamber, it is possible to calculate the theoretical percentage concentration of the medium created in the culture chamber after any given number of hours. The following mathematical relationships are used for the calculations:

where

$C_T = C_0 + (C_i - C_0)(1 - e^{-(T-T_0)(R/V)})$
C_T = outflow concentration at an arbitrary moment
C_0 = concentration at $T = T_0$
C_i = concentration of inflow
R = flow rate (ml/h)
V = volume of chamber (ml)
T = time at arbitrary moment
T_0 = starting time."

Show that you can rather easily justify this somewhat horrendous "out-of-the-magic-hat" formula.

†24. A snowfall begins sometime in the forenoon, and snows steadily on into the afternoon. At noon a man begins to clear the sidewalk on a certain street, shoveling at a constant rate (in cubic feet per hour) and at a constant width. He shovels two blocks by 2 p.m. and one block more by 4 p.m. At what time did the snow begin to fall? (You may assume he does not go back to clear the snow that has fallen behind him.)

Solutions

1. a) $dy/dt = ky$, so

$$\frac{dy}{y} = k\,dt \quad \text{(separating variables)}$$
$$\ln y = kt + C \quad \text{(integrating both sides)}$$
$$y = e^{kt+C} \quad \text{(expressing exponentially)}$$
$$= e^C e^{kt}$$
$$= A\,e^{kt} \quad \text{(renaming single constant).}$$

b) $y(24) = 100\,e^{24k} = 400$
$e^{24k} = 4$ (isolating factor with e)
$24k = \ln 4$ (taking ln both sides)
$\bullet\ k = \dfrac{\ln 4}{24}$.

c) $y(12) = 100\,e^{12\left(\frac{\ln 4}{24}\right)}$
$= 100(e^{\ln 4})^{1/2} = 100(4)^{1/2} = 200.$

d) $y = A^* e^{k^* t}$
$y(0) = 100 \Rightarrow A^* = 100$
$y(1) = 400 \Rightarrow 400 = 100\,e^{k^*} \Rightarrow e^{k^*} = 4 \Rightarrow k^* = \ln 4$
$y(\tfrac{1}{2}) = 100\,e^{(1/2)\ln 4} = 100(e^{\ln 4})^{1/2} = 100(4)^{1/2} = 200.$

2. a) From $dT/dt = kT - km$, you can see that you cannot separate T from m, but $[dT/(T - m)] + K\,dt$ works very nicely.
b) $m = 85°$.

3. a) $S(1) = S_0 + rS_0 = (1 + r)S_0$
$S(2) = S_1 + rS_1 = (1 + r)S_1 = (1 + r)^2 S_0$
$\vdots$
$S(n) = S_{n-1} + rS_{n-1} = (1 + r)S_{n-1} = (1 + r)^n S_0.$

d) $$S\left(\frac{n}{m}\right) = \left(1 + \frac{r}{m}\right)^n S_0$$

$$\lim_{m\to\infty} S\left(\frac{n}{m}\right) = \lim_{m\to\infty}\left(\underbrace{\left(1 + \frac{r}{m}\right)^m}\right)^{n/m} S_0$$
$$= \lim_{m\to\infty} (e^r)^{n/m} S_0,$$

so $S(t) = S_0\,e^{rt}$.

4. $$\frac{y_0}{2} = y_0\,e^{-t/8000}$$

$$\underbrace{\ln 0.5}_{\approx 0.7} = -t/8000$$

so $\bullet\ t \approx 5600$ yr.

5. $y/y_0 = e^{-t/8000}$. In 1868, $t = 31{,}868$ represents 30,000 B.C., $t = 21{,}868$ represents 20,000 B.C.

$$e^{-31{,}868/8000} < \text{lab.}\ \frac{y}{y_0} < e^{-21{,}868/8000}$$

$$e^{-3.9835} < \text{lab.}\ \frac{y}{y_0} < e^{-2.7335}$$

$$\bullet\ \text{approx.}\ \underbrace{0.019}_{1.9\%} < \text{lab.}\ \frac{y}{y_0} < \text{approx.}\ \underbrace{0.065}_{6.5\%}.$$

6. $T = 20 + 80(\tfrac{1}{2})^{t/20}$. $\bullet$ When $T = 30°$, $t = 60$ min. (Assuming boiling $T = 100°$ at $t = 0$, $T = 60°$ at $t = 20$.)

7. First you will have to ask your friend the room temperature R. Then $dT/dt = -k(T - R)$ which has solution $\bullet\ T - R = Ce^{-kt}$. At $t = 0$, $T = 212$, so $T - R = 212\,e^{-kt}$. At $t = ?$, $T = 185$, but you are still stuck because you do not know k. You will have to ask him for one more bit of information. Unless he can recall something like "the day I was on the phone for half an hour, the stupid water cooled down to 95° . . . ," you are probably best off fetching him a new thermometer.

8. a) After 6 min, $T = 82.5°C$.
b) at 75°, $T = 10.2$ min.
at 61°, $T = 38.5$ minutes.

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} T = 60 + 40(\tfrac{3}{4})^{t/3}.$$

9. Obviously, this is *not* a situation in which the surrounding medium remains at a constant temperature, so we need a modification of Newton's Law of Cooling as seen in Example 4. If you are not an expert in physics, all is *not* lost—think a bit about what might make *sense*. You should realize that the volume of the objects is somehow involved, for the small one (the ounce of hot water) will obviously (intuitively) cool off more than the large one (the 100-oz solution) will warm up. You can make a proposal based on this, and one way to check it will be to see if it reduces to the Newton model of Example 4 in the case where the surrounding medium is large enough to remain at constant temperature.

Heat flow is what is conserved in a situation where objects of finite volume and different temperatures meet. *Heat flow* is what is experimentally found to be linearly proportional to the temperature difference.

This means that • $\underbrace{\frac{d}{dt}V_1T_1}_{\substack{T_1 \text{ going down;} \\ \text{deriv. will be} \\ \text{negative}}} = \underbrace{-\frac{d}{dt}V_2T_2}_{\substack{T_2 \text{ going up;} \\ \text{deriv. will} \\ \text{be positive}}} = k(T_2 - T_1).$

If this did not occur to you, do you agree that it sounds reasonable? You should be able to find a convincing argument in any elementary physics text, under heat flow or heat transfer.

In this problem, $V_1 = 1$ oz and $V_2 = 100$ oz; both are constant. So we have

$$\frac{dT_1}{dt} = -100\frac{dT_2}{dt} = K(T_2 - T_1).$$

Since T_1 and T_2 are both functions of t, this is a simple *system* of two differential equations in $T_1(t)$ and $T_2(t)$. Details of solving such a *system* are beyond the scope of this chapter (though *setting up* the system is not). The answer is $T_2 = (1090/101)(1 - e^{-kt}) + 10e^{-kt}$.

10. Volume depends on r^3, surface area on r^2. Therefore, surface area depends on $V^{2/3}$; $V = k_1r^3$; $S = k_2r^2 = k_2(\sqrt[3]{V/k_1})^2 = KV^{2/3}$. Differentiating, $dV/dt = -cV^{2/3}$ (negative to show *decrease* in V). By separation of variables,

$$3V^{1/3} = -ct + Q$$

$$V = \left(\frac{-ct + Q}{3}\right)^2.$$

At $t = 0$, $V = V_0$, so • $V = \left(-\frac{c}{3}t + V_0^{1/3}\right)^3$.

11. Let S = amount of salt; rate = input − outgo.

$$\frac{dS}{dt} = \left(0\frac{\text{lb}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{S\text{ lb}}{100\text{ gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) = -\frac{S}{50}$$

$S = ke^{-(1/50)t}$ at $t = 0$, $s = 20$, so

$$S = 20e^{-0.02t}$$

$$\bullet\ S(60) = 20e^{-1.2}.$$

12. $\frac{dz}{dt} = \text{input} - \text{outgo} = \left(r_1\frac{\text{gal}}{\text{min}}\right)\left(c_1\frac{\text{lb}}{\text{gal}}\right) - \left(r_2\frac{\text{gal}}{\text{min}}\right)\left(\frac{z\text{ lb}}{V_0 + (r_1 - r_2)t\text{ gal}}\right)$

$$= r_1c_1 - \frac{r_2z}{V_0 + (r_1 - r_2)t}.$$

At $t = 0$, $z = C_0$. Use to evaluate constant of integration. The question asks for

$$\bullet\ t \text{ when } \underbrace{\frac{z}{V_0 + (r_1 - r_2)t}}_{\substack{\text{concentration} \\ \text{of effluent}}} = 0.30C.$$

13. $dS/dt = rS$, $S = S_0e^{rt}$. When savings double, $t = 16$ implies

$$2S_0 = S_0e^{16r}$$

$$2 = e^{16r}$$

$$r = \frac{\ln 2}{16} = 0.0433 = \bullet\ 4.33\%.$$

14. Both the differential equation and its solution are going to have discontinuities every time that t = an integer number of years.

$$\frac{dS}{dt} = \begin{cases} rS, & \text{for } t \neq n \\ rS + A, & \text{for } t = n \end{cases}$$

$$S(t) = S_0e^{rt} + Ae^{rt}\left(\frac{1}{e^r} + \frac{1}{e^{2r}} + \frac{1}{e^{3r}} + \cdots + \frac{1}{e^{kr}}\right)$$

where k is the greatest integer less than t and $t \geq 1$ (for $0 \leq t < 1$, $S = S_0e^{rt}$). (See Fig. 1.2).

15. S = amount of salt,

$$dS/dt = \text{input} - \text{outflow}$$

$$= \left(2\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right) - \left(\frac{S}{10}\frac{\text{lb}}{\text{gal}}\right)\left(3\frac{\text{gal}}{\text{min}}\right)$$

$$= 6 - \tfrac{3}{10}S$$

$S = 20(1 - Ce^{-(3/10)t})$; $S(0) = 5$, so $C = \frac{3}{4}$. • $S = 20(1 - (3/4)e^{-(3/10)t})$
a) After 8 min, *concentration* = $S(8)/10 = 2(1 - (3/4)e^{-2.4})$lb/gal.
b) Long term, as $t \to \infty$, $S \to 20$ lb in tank.

16. Let $N(t)$ = number of neutrons present. Then $dN/dt = kN$, so $N(t) = N_0e^{kt}$. Then

$$\left(\frac{N_2}{N_0}\right)^{T_1} = \left(\frac{N(T_2)}{N_0}\right)^{T_1} = (e^{kT_2})^{T_1} = e^{kT_1T_2} = (e^{kT_1})^{T_2} = \left(\frac{N_1}{N_0}\right)^{T_2}.$$

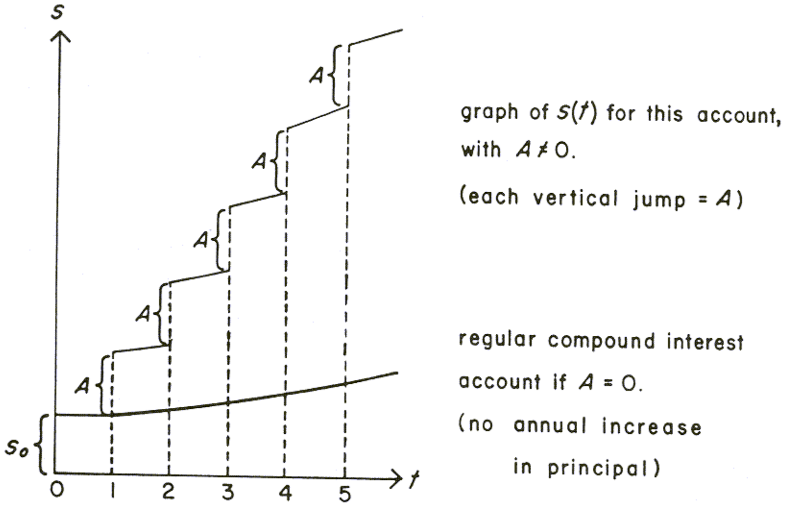

Figure 1.2. The lower curve shows regular compound interest account if $A = 0$ (no annual increase in principal). The stepped curves above show the graph of $S(t)$ for this account, with $A \neq 0$. Each vertical jump $= A$.

17. Let $P(t)$ = amount of pollutant in tank,

$$\frac{dP}{dt} = \text{input} - \text{outgo}$$

$$= \underbrace{\left(2\frac{\text{oz}}{\text{gal}}\right)\left(500\frac{\text{gal}}{\text{min}}\right)}_{\text{inflow}} - \underbrace{\left(\frac{P\text{ oz}}{10{,}000\text{ gal}}\right)\left(500\frac{\text{gal}}{\text{min}}\right)}_{\text{outflow}} - \underbrace{0.02P\frac{\text{oz}}{\text{min}}}_{\text{treatment}}.$$

So $dP/dt = 1000 - 0.07P$ and $P = (100{,}000/7)(1 - ce^{-0.07t})$. At $t = 0$, $P = 0$, so $C = 1$.

18. As in Example 8,

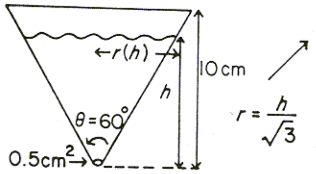

Figure 1.3

$$\underbrace{-\pi\left(\frac{h}{\sqrt{3}}\right)^2}_{\text{area of surface of water}} dh = \underbrace{0.5\sqrt{2gh}}_{\text{area of hole}}\,dt$$

$$h^{3/2}\,dh = -\frac{1.5}{\pi}\sqrt{2g}\,dt$$

$$\int_{10}^{0} h^{3/2}\,dh = -\frac{1.5}{\pi}\sqrt{2g}\int_0^T dt$$

$$\tfrac{2}{5}h^{5/2}\Big]_{10}^{0} = -\frac{1.5}{\pi}\sqrt{2g}\,t\Big]_0^T$$

$$-\tfrac{2}{5}(10)^{5/2} = -\frac{1.5}{\pi}\sqrt{2g}\,T$$

$$T = \frac{2\pi}{7.5\sqrt{2g}}(10)^{5/2}\text{ s.}$$

19. Let $y_1(t)$ and $y_2(t)$ be the amount of salt, in pounds, in the first and second vats, respectively, at time t.

First tank: $\dfrac{dy_1}{dt} = \text{input} - \text{outgo}$

$$= \left(0\frac{\text{lb}}{\text{gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y_1\text{ lb}}{100\text{ gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right)$$

$$= -\frac{y_1}{50}$$

$$y_1 = Ce^{-t/50};\ y_1(0) = 50;\ \bullet\ y_1 = 50e^{-t/50}.$$

Second tank: $\dfrac{dy_2}{dt} = \text{input} - \text{outgo}$

$$= \left(\frac{y_1\text{ lb}}{100\text{ gal}}\right)\left(2\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y_2\text{ lb}}{(100+t)\text{ gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right)$$

$$= e^{-t/50} - \frac{y_2}{100+t}.$$

Solving as linear equation with integrating factor,

$$y_2 = k - 50(150 + t)e^{-t/50}.$$

$y_2(0) = 50$, so $k = 12{,}500$. • The *concentration* of salt in water leaving second tank:

$$\frac{y_2}{100+t} = \frac{12{,}500}{100+t} - 50\left(\frac{150+t}{100+t}\right)e^{-t/50}.$$

20. $G(t)$ = concentration of glucose,
A = rate at which glucose is admitted, in mg/min,
rate = input − outgo,

$$\frac{dG}{dt} = \frac{A}{V} - KG \quad \text{(decrease proportional to } G\text{)}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \nwarrow$$

$$\frac{\text{mg/cm}^3}{\text{min}} \qquad \frac{\text{mg/min}}{\text{cm}^3} \qquad \left(\frac{1}{\text{min}}\right)\frac{\text{mg}}{\text{cm}^3}.$$

It is not necessary to *solve* this differential equation to answer the question of the problem. An equilibrium level will be a *constant* concentration G such that G no longer changes, i.e., $dG/dt = 0$.

$$\frac{dG}{dt} = \frac{A}{V} - kG = 0$$

where $G = A/KV$ is equilibrium concentration. Still without solving, you can see from the differential equation that

$$G > \frac{A}{KV} \text{ gives negative } \frac{dG}{dt} \text{ (decreasing } G\text{)}$$

$$G < \frac{A}{KV} \text{ gives positive } \frac{dG}{dt} \text{ (increasing } G\text{).}$$

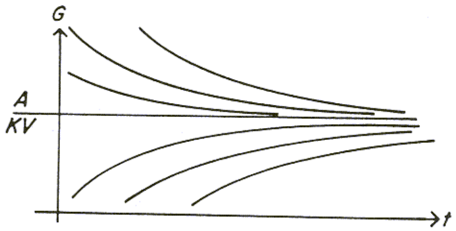

Figure 1.4

The solutions for $G(t)$ must look like those in Fig. 1.4 (with different values of the constant arising from integration corresponding to different solution curves). These solution curves are further confirmed by noting that if

$$\frac{dG}{dt} = \frac{A}{V} - KG,$$

then

$$\frac{d^2G}{dt^2} = -K\frac{dG}{dt} = -K\left(\frac{A}{V} - KG\right),$$

positive if $G > A/KV$ (then G is concave up), negative if $G < A/KV$ (then G is concave down) (Actual solution: $G = (A/KV)(1 - e^{-kt})$.)

21. Possible variable volume model: V is no longer constant, S is the volume of solution per minute being infused;

$$\bullet \quad \underset{\text{cm}^3}{V} = \underset{\text{cm}^3}{V_0} + \underset{\left(\frac{\text{cm}^2}{\text{min}}\right)\text{min}}{St.}$$

There is a relation between S and A:

$$\bullet \quad \frac{A}{S} = \left(\frac{\text{mg/min}}{\text{cm}^3/\text{min}}\right)\text{glucose solution} = \frac{\text{mg}}{\text{cm}^3} = C,$$

is the constant concentration of glucose in solution being infused. So $V = V_0 + (1/c)At$, and the differential equation of Exercise 20 becomes

$$\bullet \quad \frac{dG}{dt} = \frac{A}{V_0 + \frac{A}{C}t} - kG.$$

There now is *no* equilibrium level of G because no *constant* G produces $dG/dt = 0$.
This model appears messy either to solve *or* to sketch solutions as we did in Exercise 20. As with many differential equations, a numerical approach to a

solution seems the only sensible one, but at this point you might easily question the value of the whole variable volume consideration, since it has not produced any immediate physical insights.

Thinking about variable volume further, you would realize that B cannot physically increase this way for very long—the body does not have unlimited capacity for increasing volume of liquid in the veins. Other mechanisms such as urination will work to keep the volume in line, so you would want to ask the medical and biological experts about the realistic limits of variation in volume; how long a glucose solution may be steadily infused; whether an equilibrium concentration is actually observed, etc. One reference on this topic which provides some actual data is Defares and Sneddon, *The Mathematics of Medicine and Biology* (Chicago: 1961).[3]

22. Let y be the concentration of B and A_0 be the initial concentration of A:

$$\frac{dy}{dt} = c(A_0 - y) \qquad y(0) = 0;\ y(20) = \frac{A_0}{2}$$

$$\underbrace{-\ln|A_0 - y|}_{\text{always positive}} = ct + K$$

$$A_0 - y = Qe^{-ct} \qquad Q = A_0;\ e^{-20c} = \tfrac{1}{2}$$

$$\text{so } c = \frac{\ln 2}{20}$$

$$\bullet \quad y = A_0(1 - (\tfrac{1}{2})^{t/20})$$

$$y' = \underset{\text{pos.}}{c}(\underset{\text{pos.}}{A_0 - y}) \quad \text{always positive}$$

$$y'' = -cy' = \underbrace{\underbrace{-c^2}_{\text{neg.}}\underbrace{(A_0 - y)}_{\text{pos.}}}_{\text{always negative; } y \text{ concave down}}$$

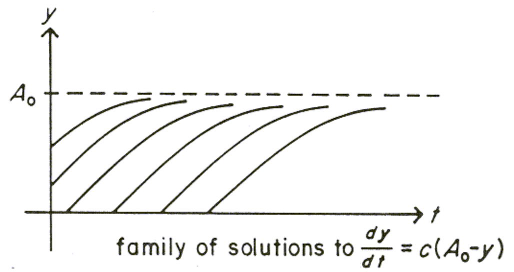

Figure 1.5

The family of solutions to $y' = c(A_0 - y)$ is shown in Fig. 1.5.

[3] They refer to S. G. Jokipii *et al.*, *J. Clin. Invest.*, vol. 34, 1954, pp. 331, 452, 458.

23. C_T = concentration in the outflow,
= concentration of the medium in the vessel at any moment T.
Therefore, VC_T = *amount* of the medium in the vessel at T.
Rate of change = input − outflow

$$\frac{d}{dt}(VC_T) = \left(R\frac{\text{ml}}{\text{h}}\right)\left(C_i\frac{\text{mg}}{\text{ml}}\right) - \left(R\frac{\text{ml}}{\text{h}}\right)\left(\frac{VC_T\ \text{mg}}{V\ \text{ml}}\right).$$

V, R, and C_i are all constant, so

$$V\frac{d}{dt}C_T = RC_i - RC_T$$

or

$$\bullet\ \frac{d}{dt}C_T = \frac{R}{V}(C_i - C_T).$$

Solution by separation of variables (the use of $T - T_0$ in the "formula" is a clue that definite integrals may lead us most quickly to the result):

$$\int_{C_0}^{C_T} \frac{dC_T}{C_i\left(1 - \frac{1}{C_i}C_T\right)} = \int_{T_0}^{T} \frac{R}{V}\,dt$$

$$-\ln\left|1 - \frac{C_T}{C_i}\right|\Bigg]_{C_0}^{C_T} = \frac{R}{V}t\Bigg]_{T_0}^{T}$$

$$\ln\left|1 - \frac{C_T}{C_i}\right| - \ln\left|1 - \frac{C_0}{C_i}\right| = -\frac{R}{V}(T - T_0)$$

$$\frac{\left|1 - \frac{C_T}{C_i}\right|}{\left|1 - \frac{C_0}{C_i}\right|} = \exp\left[-\frac{R}{V}(T - T_0)\right].$$

Because of the absolute value signs, the resulting formula is only valid if $C_T > C_i$ *and* $C_0 < C_i$ or if $C_T < C_i$ *and* $C_0 < C_i$. This does not seem to be *required* of the chemostat, but perhaps in practice it is the case.

$$C_i - C_T = (C_i - C_0)\exp\left[-\frac{R}{V}(T - T_0)\right]$$

$$C_T = C_i - (C_i - C_0)\exp\left[-\frac{R}{V}(T - T_0)\right]$$

which can indeed be shown to equal

$$C_0 + (C_i - C_0)\left(1 - \exp\left[-\frac{R}{V}(T - T_0)\right]\right).$$

24. *Hint:* Look at the *velocity* at which the shoveler *travels*, which is inversely proportional to the volume of snow that has fallen on any spot at any instant. Answer: $\sqrt{5} - 1$ h before noon.

Notes for the Instructor

Objectives. Applied problems requiring differential equations seem to be harder for many students to translate into mathematical terms than problems met heretofore. Extra attention is needed to the following facts.

(1) The differential equation is an *instantaneous* statement, which must be valid *at any time.*
(2) Not all the numbers given in a problem go into the differential equation. Some must be held out to provide the conditions necessary to evaluate the constants of integration, or other parameters, such as those of proportionality.
(3) Matching physical dimensions for each term of the differential equation is essential and not usually automatic.
(4) An organized and clearly written framework is helpful to the student and necessary for those who read his work.

Prerequisites. Elementary calculus, specifically

(1) familiarity with differentiation and basic integration,
(2) some experience with simple differential equations and their solutions.

Time. Part of one class hour should suffice for the introduction of a few varied examples. A later recitation, *after* students have worked on the exercises, can discuss those problems where they have encountered difficulties.

Remark. This module has been developed within our third semester course in calculus and differential equations which attends to the *applicability* of the mathematics. We have written especially for students who may be majoring in biological or social sciences and who may not feel agile with mathematics or physics. The module was written primarily for independent use by the students. It has focused on *setting up* the mathematical models, *not* on the subsequent solutions; some more difficult exercises (denoted by dagger) are included for this purpose.