Barrow's Proof of the Fundamental Theorem of Calculus

Barrow provides a geometric proof of the following theorem:

Theorem 0.1 *Suppose that $g(x)$ is continuous and strictly increasing for $0 \leq x < +\infty$ and $g(0) = 0$. Let $f(x) = \int_0^x g(t)dt$. Then $f'(x) = g(x)$.*

Consider the diagram below. Above the x-axis we have the curve OP, which is the graph of the function $y = f(x)$ and below the x-axis the curve OQ, the graph of $y = -g(x)$. That is, the length of the segment XP is the area between the x-axis and the graph from O to Q. Pick point T so that the slope of line PT is equal to the length XQ; in other words, the slope of the line PT equals $g(X)$. To prove the theorem we need to show that the line PT is tangent to the upper graph at the point P.

Select R on the curve OP and S on PT with RS parallel to the x-axis. We show that R and S are distinct points, so that the line PT cannot intersect the curve OP more than once.

Triangle PSU is similar to PTX so $PU/SU = PX/SX = XQ$. Therefore, $PU = SU \cdot XQ$.

Since $UX = VR$ is the area of sector OVW and PX is the area of sector OXQ we have that PU, which is the area of region $VWQX$, is less than the area of region $VZQX$, which equals $RU \cdot XQ$. Therefore, $PU < RU \cdot XQ$, while $PU = SU \cdot XQ$. It follows that $RU > SU$ and R and S are distinct. This completes the proof.