Logarithms and Exponential functions (from Landau's Differential and Integral Calculus)

def. 1. For n=0,1,2,..., $k=2^n$ and x>0 set $a(n,x)=k((x)^{1/k}-1)$.

Theorem 1. The limit of {a(n,x)} as n goes to infinity exists; call it L(x).
proof: 1) Let x>1. We show {a(n,x)} is decreasing. Let $y=(x)^{1/2k}$. Then we have $a(n,x)= k(y^2-1)=k(y+1)(y-1)>2k(y-1)=2k((x)^{1/2k}-1)=a(n+1,x)$. The sequence {a(n,x)} is then a decreasing sequence of positive terms so converges; note that $L(x)\geq 0$. 2) For x=1 a(n,1)=0 for all n. 3) For x<0 we use $a(n,x)=-a(n,1/x)(x)^{1/k}$; now $L(x)\leq 0$.

def.2. log(x)= L(x) for all x>0.

Theorem 2. log(xy) = log(x)+log(y).
proof. $a(n,xy)=k((xy)^{1/k}-1)=k((xy)^{1/k}-(y)^{1/k}+(y)^{1/k}-1)$.

Theorem 3. $1-1/x \leq \log(x) \leq x-1$.
proof. Let $y=(x)^{1/k}$. Then $y^k-1=(y-1)(1+y+y^2+...+y^{k-1})\geq k(y-1)$.

Theorem 4. log(x) < log(y) if 0<x<y.
proof. Since x/y<1, $\log(x/y)\leq 0$. But $\log(u)\leq u-1$ and 0<u<1 implies log(u)<0.

Theorem 5. For every x the equation log(y)=x has exactly one solution.
proof. By theorem 4 there is at most one solution. Without loss of generality, let x>0. Then, for each z say z is in Class 1 if $z\leq 0$ or if $\log(z)\leq x$; say z is in Class 2 if log(z)>x. Then every z is in precisely one class. Class 1 contains z=1. Class 2 contains $z=2^m$ if m log(2)>x. If z is in Class 2 and w>z then w is also in Class 2. Consequently, there must be a y such that z<y implies that z is in Class 1 and z>y implies that z is in Class 2. Then we show that log(y)=x. Suppose not. First, suppose that log(y)<x. Set h=x-log(y), z=(1+h)y and show that $\log(z)\leq x$ (contradiction). Second, suppose that log(y)>x. Set h=[log(y)-x]/2 and z=y/(1+h). Then $\log(z)\geq x$; but this is also a contradiction. So, log(y)=x.

def.3. y=e solves log(y)=1.

Note: For any rational x, $y= e^x$ solves log(y)=x. This motivates the definition of $y=e^x$, for irrational x, as the solution of the equation log(y)=x.

We can now establish all the usual algebraic properties of these functions; the details are omitted.

Theorem 6. The limit, as x goes to zero, of (log(1+x))/x is one.
proof. For x>-1 we have $x/(1+x)\leq\log(1+x)\leq x$, so, for x>0, we have the inequalities $-x/(1+x)\leq(\log(1+x))/x-1\leq 0$. For -1<x<0 we have $1/(1+x)\geq(\log(1+x))/x\geq 1$, so that $-x/(1+x)\geq(\log(1+x))/x-1\geq 0$. Consequently, for $0<|x|\leq 1/2$, $|(\log(1+x))/x-1|<2|x|$.

Theorem 7. d/dx(log(x))=1/x .
proof. Let g(x)=log(x). Then (1/h)(g(x+h)-g(x))=(1/x)[(log(1+h/x))/(h/x)].

Theorem 8. $d/dx(e^x)=e^x$.

proof. From theorem 6 we have that the limit, as x goes to one, of (x-1)/log(x) is one. Therefore, the limit, as h goes to zero, of $[e^h-1]/h$ is one.