# Examples Illustrating the Algebra of Matrices

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## 1 Introduction

There are three basic algebraic operations that we perform on matrices: 1) addition; 2) multiplication of one matrix by another compatible one and 3) multiplication of a matrix by a scalar. The second of these, at least, may seem a bit unmotivated. The purpose of this note is to illustrate the usefulness of these three operations through examples. The examples are but two of many interesting ones discussed in the book by Anton and Rorres [1].

## 2 Family Influence

Our first example concerns family influence. An arrow from A to B in the graph below indicates that A can influence B directly. If the daughter (D) wants the father's (F) permission to stay out late, she just asks him, confident in getting her way. If the older son (OS), on the other hand, wants the father to raise his allowance he must ask the younger son (YS) to talk to the mother (M), to get her to talk to the daughter (D) and convince her to approach the father (F).

We make all this a bit more mathematical using a matrix, that is, a two dimensional array of numbers. The influence matrix obtained from the graph is

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The horizontal rows correspond to M,F,D,OS and YS, in that order from top to bottom and the columns are in the same order from left to right. The ones in the third and fourth columns of row one indicate that the mother can influence directly both the daughter and older son, as the graph tells us.

Suppose the daughter wants to get the younger son to stop bringing his friends home after school. She can't convince him directly, so she must take another route. Studying the graph is one way to figure out what she must do, but there is another way.

The daughter computes the following sum of products:

$$M_{31}M_{15} + M_{32}M_{25} + M_{33}M_{35} + M_{34}M_{45} + M_{35}M_{55}. \tag{2.1}$$

where $M_{mn}$ indicates the number in the $m$th row and $n$th column of matrix $M$. If any one of the products in (2.1) is one (for example, $M_{32}M_{25} = 1$) that means that both factors are nonzero , so that the daughter can influence the younger son using the person with the third index (2 in this case). The sum of products we just formed helps us define the product of two matrices. In this case we shall multiply $M$ by $M$ to get $M^2$. For $m = 1, ..., 5$ and $n = 1, ..., 5$ the entry in the $m$th row and $n$th column of $M^2$ will be

$$(M^2)_{mn} = \sum_{k=1}^{5} M_{mk}M_{kn}.$$

Performing these calculations (or rather, getting MATLAB to do it) we find

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The matrix $M^2$ tells us who can influence whom with the help of exactly one other person. Suppose now that we add $M$ and $M^2$, which means form the matrix $M + M^2$ by adding together the entries in the same row and column of both. Then we find that

$$M + M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Notice that the entry in the fourth row and second column is still zero, meaning that the older son can't influence the father directly or by using the help of one other person! Compute the matrix $M + M^2 + M^3$ and see how the older son is doing in his efforts. What about $M + M^2 + M^3 + M^4$?

## 3 Which Team is the Best?

Suppose now that each of five baseball teams plays all the other four and we indicate the results using a matrix $W$, with a one in the $m$th row and $n$th column if team $m$ beat team $n$ when they met. Suppose the matrix $W$ is

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

When we add across each row we find that teams 2 and 5 both had three victories and everyone else had fewer than three. So who is the best team? We know that team 2 beat team 5 when they met, so perhaps 2 is the best. But 5 beat 4 and 4 beat 2, so maybe that should count for something. Once again, we compute $W + W^2$ to get

$$W + W^2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 3 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 & 0 \end{bmatrix}$$

Now team 2 has a row sum of nine, while team 5 has a row sum of seven; even taking into consideration the results involving a third team we find that 2 is still the best. But wait! Who says we have to count the matrix $W^2$ as heavily as the matrix $W$? Shouldn't the head-to-head matches mean more than these other three-way combinations? OK! Suppose we agree to weight head-to-head twice as much as three-way combinations. Let's calculate $\frac{2}{3}W + \frac{1}{3}W^2$:

$$\frac{2}{3}W + \frac{1}{3}W^2 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ 1 & 0 & \frac{4}{3} & 1 & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & 0\frac{1}{3} & 0 & \frac{1}{3} \\ 1\frac{2}{3} & \frac{1}{3} & 1 & \frac{4}{3} & 0 \end{bmatrix}$$

Now who has the largest row sum?

## References


[1] Anton, H., and Rorres, C. (1987) *Elementary Linear Algebra with Applications*, Wiley.

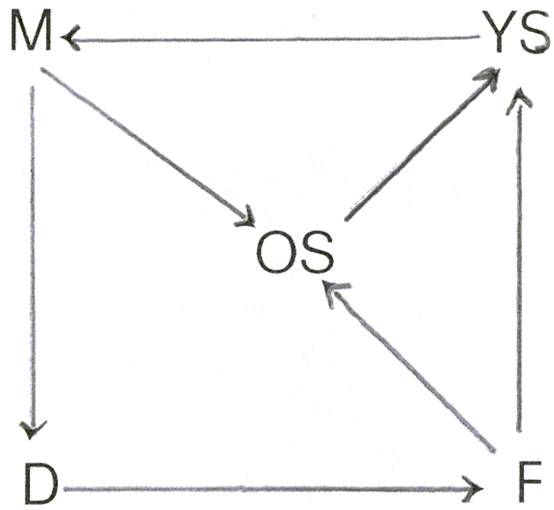


Figure 1: Family Influence Graph