A First Course in Optimization: Answers to Selected Exercises

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Chapter 1

Preface

In the chapters that follow you will find solutions to many, but not all, of the exercises in the text. Some chapters in the text have no exercises, but those chapters are included here to keep the chapter numbers the same as in the text. Please note that the numbering of exercises within each chapter may differ from the numbering in the text itself, because certain exercises have been skipped.

Chapter 2

Introduction

2.1 *Consider the function $f(x)$ defined by $f(x) = e^{-x}$, for $x > 0$ and by $f(x) = -e^x$, for $x < 0$. Show that*

$$-1 = \liminf_{x \to 0} f(x)$$

and

$$1 = \limsup_{x \to 0} f(x).$$

As we approach $x = 0$ from the left, the limit is -1, while as we approach from the right, the limit is $+1$. Therefore, the set S is $S = \{-1, 0, 1\}$. The greatest lower bound for S is its minimum member, -1, while the least upper bound is its maximum member, $+1$.

2.1 Exercise

2.2 *For $n = 1, 2, ...,$ let*

$$A_n = \{x | \; \|x - a\| \leq \frac{1}{n}\},$$

and let α_n and β_n be defined by

$$\alpha_n = \inf\{f(x) | \, x \in A_n\},$$

and

$$\beta_n = \sup\{f(x) | \, x \in A_n\}.$$

- *a) Show that the sequence $\{\alpha_n\}$ is increasing, bounded above by $f(a)$ and converges to some α, while the sequence $\{\beta_n\}$ is decreasing, bounded below by $f(a)$ and converges to some β. Hint: use the fact that, if $A \subseteq B$, where A and B are sets of real numbers, then* $\inf(A) \geq \inf(B)$.

- *b) Show that* α *and* β *are in* S*. Hint: prove that there is a sequence* $\{x^n\}$ *with* x^n *in* A_n *and* $f(x^n) \leq \alpha_n + \frac{1}{n}$*.*

- *c) Show that, if* $\{x^m\}$ *is any sequence converging to* a*, then there is a subsequence, denoted* $\{x^{m_n}\}$*, such that* x^{m_n} *is in* A_n*, for each* n*, and so*
$$\alpha_n \leq f(x^{m_n}) \leq \beta_n.$$

- *d) Show that, if* $\{f(x^m)\}$ *converges to* γ*, then*
$$\alpha \leq \gamma \leq \beta.$$

- *e) Show that*
$$\alpha = \liminf_{x \to a} f(x)$$
and
$$\beta = \limsup_{x \to a} f(x).$$

According to the hint in a), we use the fact that $A_{n+1} \subseteq A_n$. For b), since α_n is the infimum, there must be a member of A_n, call it x^n, such that
$$\alpha_n \leq f(x^n) \leq \alpha_n + \frac{1}{n}.$$
Then the sequence $\{x^n\}$ converges to a and the sequence $\{f(x^n)\}$ converges to α.

Now suppose that $\{x^m\}$ converges to a and $\{f(x^m)\}$ converges to γ. We don't know that each x^m is in A_m, but we do know that, for each n, there must be a term of the sequence, call it x^{m_n}, that is within $\frac{1}{n}$ of a, that is, x^{m_n} is in A_n. From c) we have
$$\alpha_n \leq f(x^{m_n}) \leq \beta_n.$$
The sequence $\{f(x^{m_n})\}$ also converges to γ, so that
$$\alpha \leq \gamma \leq \beta.$$
Finally, since we have shown that α is the smallest number in S it follows that
$$\alpha = \liminf_{x \to a} f(x).$$
The argument for β is similar.

Chapter 3

Optimization Without Calculus

3.1 Exercises

3.1 *Let A be the arithmetic mean of a finite set of positive numbers, with x the smallest of these numbers, and y the largest. Show that*

$$xy \leq A(x + y - A),$$

with equality if and only if $x = y = A$.

This is equivalent to showing that

$$A^2 - A(x + y) + xy = (A - x)(A - y) \leq 0,$$

which is true, since $A - x \geq 0$ and $A - y \leq 0$.

3.2 *Minimize the function*

$$f(x) = x^2 + \frac{1}{x^2} + 4x + \frac{4}{x},$$

over positive x. Note that the minimum value of $f(x, y)$ cannot be found by a straight-forward application of the AGM Inequality to the four terms taken together. Try to find a way of rewriting $f(x)$, perhaps using more than four terms, so that the AGM Inequality can be applied to all the terms.

The product of the four terms is 16, whose fourth root is 2, so the AGM Inequality tells us that $f(x) \geq 8$, with equality if and only if all four terms are equal. But it is not possible for all four terms to be equal. Instead, we can write

$$f(x) = x^2 + \frac{1}{x^2} + x + \frac{1}{x} + x + \frac{1}{x} + x + \frac{1}{x} + x + \frac{1}{x}.$$

These ten terms have a product of 1, which tells us that

$$f(x) \geq 10,$$

with equality if and only if $x = 1$.

3.3 *Find the maximum value of* $f(x, y) = x^2 y$, *if* x *and* y *are restricted to positive real numbers for which* $6x + 5y = 45$.

Write

$$f(x, y) = xxy = \frac{1}{45}(3x)(3x)(5y),$$

and

$$45 = 6x + 5y = 3x + 3x + 5y.$$

3.4 *Find the smallest value of*

$$f(x) = 5x + \frac{16}{x} + 21,$$

over positive x.

Just focus on the first two terms.

3.5 *Find the smallest value of the function*

$$f(x) = \sqrt{x^2 + y^2},$$

among those values of x *and* y *satisfying* $3x - y = 20$.

By Cauchy's Inequality, we know that

$$20 = 3x - y = (x, y) \cdot (3, -1) \leq \sqrt{x^2 + y^2}\sqrt{3^2 + (-1)^2} = \sqrt{10}\sqrt{x^2 + y^2},$$

with equality if and only if $(x, y) = c(3, -1)$ for some number c. Then solve for c.

3.6 *Find the maximum and minimum values of the function*

$$f(x) = \sqrt{100 + x^2} - x$$

over non-negative x.

The derivative of $f(x)$ is negative, so $f(x)$ is decreasing and attains its maximum of 10 at $x = 0$. As $x \to \infty$, $f(x)$ goes to zero, but never reaches zero.

3.7 *Multiply out the product*

$$(x + y + z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$

and deduce that the least value of this product, over non-negative x, y, and z, is 9. Use this to find the least value of the function

$$f(x) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

over non-negative x, y, and z having a constant sum c.

See the more general Exercise 3.9.

3.8 *The* harmonic mean *of positive numbers $a_1, ..., a_N$ is*

$$H = [(\frac{1}{a_1} + ... + \frac{1}{a_N})/N]^{-1}.$$

Prove that the geometric mean G is not less than H.

Use the AGM Inequality to show that $H^{-1} \geq G^{-1}$.

3.9 *Prove that*

$$(\frac{1}{a_1} + ... + \frac{1}{a_N})(a_1 + ... + a_N) \geq N^2,$$

with equality if and only if $a_1 = ... = a_N$.

When we multiply everything out, we find that there are N ones, and $\frac{N(N-1)}{2}$ pairs of the form $\frac{a_m}{a_n} + \frac{a_n}{a_m}$, for $m \neq n$. Each of these pairs has the form $x + \frac{1}{x}$, which is always greater than or equal to two. Therefore, the entire product is greater than or equal to $N + N(N - 1) = N^2$, with equality if and only if all the a_n are the same.

3.10 *Show that the Equation $S = ULU^T$, can be written as*

$$S = \lambda_1 u^1 (u^1)^T + \lambda_2 u^2 (u^2)^T + ... + \lambda_N u^N (u^N)^T, \tag{3.1}$$

and

$$S^{-1} = \frac{1}{\lambda_1} u^1 (u^1)^T + \frac{1}{\lambda_2} u^2 (u^2)^T + ... + \frac{1}{\lambda_N} u^N (u^N)^T. \tag{3.2}$$

This is just algebra.

3.11 *Let Q be positive-definite, with positive eigenvalues*

$$\lambda_1 \geq ... \geq \lambda_N > 0$$

and associated mutually orthogonal norm-one eigenvectors u^n. Show that

$$x^T Q x \leq \lambda_1,$$

for all vectors x with $\|x\| = 1$, with equality if $x = u^1$. Hints: use

$$1 = \|x\|^2 = x^T x = x^T I x,$$

$$I = u^1(u^1)^T + ... + u^N(u^N)^T,$$

and Equation (3.1).

Use the inequality

$$x^T Q x = \sum_{n=1}^{N} \lambda_n |x^T u^n|^2 \leq \lambda_1 \sum_{n=1}^{N} |x^T u^n|^2 = \lambda_1.$$

3.12 *Relate Example 4 to eigenvectors and eigenvalues.*

We can write

$$\begin{bmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{bmatrix} = 49(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})^T(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}),$$

so that

$$(2x + 3y + 6z)^2 = |(x, y, z)(2, 3, 6)^T|^2 = (x, y, z) \begin{bmatrix} 4 & 6 & 12 \\ 6 & 9 & 18 \\ 12 & 18 & 36 \end{bmatrix} (x, y, z)^T.$$

The only positive eigenvalue of the matrix is 7 and the corresponding normalized eigenvector is $(\frac{2}{7}, \frac{3}{7}, \frac{6}{7})^T$. Then use Exercise 3.11.

3.13 Young's Inequality *Suppose that p and q are positive numbers greater than one such that $\frac{1}{p} + \frac{1}{q} = 1$. If x and y are positive numbers, then*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q},$$

with equality if and only if $x^p = y^q$. Hint: use the GAGM Inequality.

This one is pretty easy.

3.14 *For given constants c and d, find the largest and smallest values of* $cx + dy$ *taken over all points* (x, y) *of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

3.15 *Find the largest and smallest values of* $2x+y$ *on the circle* $x^2+y^2 = 1$. *Where do these values occur? What does this have to do with eigenvectors and eigenvalues?*

This one is similar to Exercise 3.12.

3.16 *When a real* M *by* N *matrix* A *is stored in the computer it is usually vectorized; that is, the matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ . & & & \\ . & & & \\ . & & & \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{bmatrix}$$

becomes

$$\mathbf{vec}(A) = (A_{11}, A_{21}, ..., A_{M1}, A_{12}, A_{22}, ..., A_{M2}, ..., A_{MN})^T.$$

Show that the dot product $\mathbf{vec}(A) \cdot \mathbf{vec}(B) = \mathbf{vec}(B)^T \mathbf{vec}(A)$ *can be obtained by*

$$\mathbf{vec}(A) \cdot \mathbf{vec}(B) = \text{trace}\,(AB^T) = \text{trace}\,(B^T A).$$

Easy.

Chapter 4

Geometric Programming

4.1 Exercise

4.1 *Show that there is no solution to the problem of minimizing the function*

$$g(t_1, t_2) = \frac{2}{t_1 t_2} + t_1 t_2 + t_1, \tag{4.1}$$

over $t_1 > 0$, $t_2 > 0$. *Can* $g(t_1, t_2)$ *ever be smaller than* $2\sqrt{2}$*?*

Show that the only vector δ that works must have $\delta_3 = 0$, which is not allowed.

To answer the second part, begin by showing that if $t_2 \leq 1$ then $g(t_1, t_2) \geq 4$, and if $t_1 \geq 1$, then $g(t_1, t_2) \geq 3$. We want to consider the case in which $t_1 \to 0$, $t_2 \to \infty$, and both $t_1 t_2$ and $(t_1 t_2)^{-1}$ remain bounded. For example, we try

$$t_2 = \frac{1}{t_1}(ae^{-t_1} + 1),$$

for some positive a. Then $g(t_1, t_2) \to \frac{2}{a+1} + a + 1$ as $t_1 \to 0$. Which a gives the smallest value? Minimizing the limit, with respect to a, gives $a = \sqrt{2} - 1$, for which the limit is $2\sqrt{2}$.

More generally, let

$$t_2 = \frac{f(t_1)}{t_1},$$

for some positive function $f(t_1)$ such that $f(t_1)$ does not go to zero as t_1 goes to zero. Then we have

$$g(t_1, t_2) = \frac{2}{f(t_1)} + f(t_1) + t_1,$$

which converges to $\frac{2}{f(0)} + f(0)$, as $t_1 \to 0$. The minimum of this limit, as a function of $f(0)$, occurs when $f(0) = \sqrt{2}$, and the minimum limit is $2\sqrt{2}$.

Chapter 5

Convex Sets

5.1 References Used in the Exercises

Proposition 5.1 *The convex hull of a set S is the set C of all convex combinations of members of S.*

Proposition 5.2 *For a given x, a vector z in C is $P_C x$ if and only if*

$$\langle c - z, z - x\rangle \geq 0, \tag{5.1}$$

for all c in the set C.

Lemma 5.1 *For $H = H(a, \gamma)$, $z = P_H x$ is the vector*

$$z = P_H x = x + (\gamma - \langle a, x\rangle)a. \tag{5.2}$$

Lemma 5.2 *For $H = H(a, \gamma)$, $H_0 = H(a, 0)$, and any x and y in R^J, we have*

$$P_H(x + y) = P_H x + P_H y - P_H 0, \tag{5.3}$$

so that

$$P_{H_0}(x + y) = P_{H_0} x + P_{H_0} y, \tag{5.4}$$

that is, the operator P_{H_0} is an additive operator. In addition,

$$P_{H_0}(\alpha x) = \alpha P_{H_0} x, \tag{5.5}$$

so that P_{H_0} is a linear operator.

Lemma 5.3 *For any hyperplane $H = H(a, \gamma)$ and $H_0 = H(a, 0)$,*

$$P_H x = P_{H_0} x + P_H 0, \tag{5.6}$$

so P_H is an affine linear operator.

Lemma 5.4 *For $i = 1, ..., I$ let H_i be the hyperplane $H_i = H(a^i, \gamma_i)$, $H_{i0} = H(a^i, 0)$, and P_i and P_{i0} the orthogonal projections onto H_i and H_{i0}, respectively. Let T be the operator $T = P_I P_{I-1} \cdots P_2 P_1$. Then $Tx = Bx + d$, for some square matrix B and vector d; that is, T is an affine linear operator.*

Definition 5.1 *Let S be a subset of R^J and $f : S \to [-\infty, \infty]$ a function defined on S. The subset of R^{J+1} defined by*

$$\text{epi}(f) = \{(x, \gamma) | f(x) \leq \gamma\}$$

is the epi-graph *of f. Then we say that f is* convex *if its epi-graph is a convex set.*

5.2 Exercises

5.1 *Let $C \subseteq R^J$, and let x^n, $n = 1, ..., N$ be members of C. For $n = 1, ..., N$, let $\alpha_n > 0$, with $\alpha_1 + ... + \alpha_N = 1$. Show that, if C is convex, then the convex combination*

$$\alpha_1 x^1 + \alpha_2 x^2 + ... + \alpha_N x^N$$

is in C.

We know that the convex combination of any two members of C is again in C. We prove the apparently more general result by induction. Suppose that it is true that any convex combination of $N - 1$ or fewer members of C is again in C. Now we show it is true for N members of C. To see this, merely write

$$\sum_{n=1}^{N} \alpha_n x^n = (1 - \alpha_N) \sum_{n=1}^{N-1} \frac{\alpha_n}{1 - \alpha_N} x^n + \alpha_N x^N.$$

5.2 *Prove Proposition 5.1. Hint: show that the set C is convex.*

Note that it is not obvious that C is convex; we need to prove this fact. We need to show that if we take two convex combinations of members of S, say x and y, and form $z = (1 - \alpha)x + \alpha y$, then z is also a member of C. To be concrete, let

$$x = \sum_{n=1}^{N} \beta_n x^n,$$

and

$$y = \sum_{m=1}^{M} \gamma_m y^m,$$

where the x^n and y^m are in S and the positive β_n and γ_m both sum to one. Then

$$z = \sum_{n=1}^{N} (1-\alpha)\beta_n x^n + \sum_{m=1}^{M} \alpha\gamma_m y^m,$$

which is a convex combination of $N + M$ members of S, since

$$\sum_{n=1}^{N} (1-\alpha)\beta_n + \sum_{m=1}^{M} \alpha\gamma_m = 1.$$

5.3 *Show that the subset of R^J consisting of all vectors x with $||x||_2 = 1$ is not convex.*

Don't make the problem harder than it is; all we need is one counter-example. Take x and $-x$, both with norm one. Then $\frac{1}{2}(x + (-x)) = 0$.

5.4 *Let $\|x\|_2 = \|y\|_2 = 1$ and $z = \frac{1}{2}(x + y)$ in R^J. Show that $\|z\|_2 < 1$ unless $x = y$. Show that this conclusion does not hold if the* two-norm $\|\cdot\|_2$ *is replaced by the* one-norm*, defined by*

$$\|x\|_1 = \sum_{j=1}^{J} |x_j|.$$

Now x and y are given, and z is their midpoint. From the Parallelogram Law,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

we have

$$\|x + y\|^2 + \|x - y\|^2 = 4,$$

so that

$$\|z\|^2 + \frac{1}{4}\|x - y\|^2 = 1,$$

from which we conclude that $\|z\| < 1$, unless $x = y$.

5.5 *Let C be the set of all vectors x in R^J with $\|x\|_2 \leq 1$. Let K be a subset of C obtained by removing from C any number of its members for which $\|x\|_2 = 1$. Show that K is convex. Consequently, every x in C with $\|x\|_2 = 1$ is an extreme point of C.*

Suppose we remove some x with $\|x\| = 1$. In order for the set without x to fail to be convex it is necessary that there be some $y \neq x$ and $z \neq x$ in C for which $x = \frac{y+z}{2}$. But we know that if $\|y\| = \|z\| = 1$, then $\|x\| < 1$, unless $y = z$, in which case we would also have $y = z = x$. So we cannot have $\|y\| = \|z\| = 1$. But what if their norms are less than one? In that case, the norm of the midpoint of y and z is not greater than the larger of the two norms, so again, it cannot be one. So no x on the boundary of C is a convex combination of two distinct members of C, and removing any number of such points leaves the remaining set convex.

5.6 *Prove that every subspace of R^J is convex, and every linear manifold is convex.*

For the first part, replace β in the definition of a subspace with $1-\alpha$, and require that $0 \leq \alpha \leq 1$. For the second part, let $M = S + b$, where S is a subspace, and b is a fixed vector. Let $x = s + b$ and $y = t + b$, where s and t are in S. Then

$$(1-\alpha)x + \alpha y = [(1-\alpha)s + \alpha t] + b,$$

which is then also in M.

5.7 *Prove that every hyperplane $H(a, \gamma)$ is a linear manifold.*

Show that

$$H(a, \gamma) = H(a, 0) + \frac{\gamma}{\|a\|^2} a.$$

5.8 *(a) Let C be a circular region in R^2. Determine the normal cone for a point on its circumference. (b) Let C be a rectangular region in R^2. Determine the normal cone for a point on its boundary.*

For (a), let x^0 be the center of the circle and x be the point on its circumference. Then the normal cone at x is the set of all z of the form $z = x + \gamma(x - x^0)$, for $\gamma \geq 0$. For (b), if the point x lies on one of the sides of the rectangle, then the normal cone consists of the points along the line segment through x formed by the outward normal to the side. If the point x is a corner, then the normal cone consists of all points in the intersection of the half-spaces external to the rectangle whose boundary lines are the two sides that meet at x.

5.9 *Prove Lemmas 5.2, 5.3 and 5.4.*

Applying Equation (5.2),

$$P_H x = x + (\gamma - \langle a, x\rangle)a,$$

we find that

$$P_H(x+y) = x+y+(\gamma-\langle a, x+y\rangle)a = x+(\gamma-\langle a, x\rangle)a+y+(\gamma-\langle a, y\rangle)a-\gamma a$$

$$= P_H(x) + P_H(y) - P_H(0).$$

We also have

$$P_H(x) = x + \gamma a - \langle a, x\rangle a = \gamma a + x - \langle a, x\rangle a = P_H(0) + P_{H_0}(x).$$

Finally, consider the case of $I = 2$. Then we have

$$Tx = P_2P_1(x) = P_2(x + (\gamma_1 - \langle a^1, x\rangle)a^1)$$

$$= [x - \langle a^1, x\rangle a^1 - \langle a^2, x\rangle a^2 + \langle a^1, x\rangle\langle a^2, a^1\rangle a^2] + \gamma_1 a^1 + \gamma_2 a^2 - \gamma_1\langle a^2, a^1\rangle a^2$$

$$= Lx + b,$$

where L is the linear operator

$$Lx = x - \langle a^1, x\rangle a^1 - \langle a^2, x\rangle a^2 + \langle a^1, x\rangle\langle a^2, a^1\rangle a^2,$$

and

$$b = \gamma_1 a^1 + \gamma_2 a^2 - \gamma_1\langle a^2, a^1\rangle a^2.$$

We can then associate with the linear operator L the matrix

$$B = I - a^1(a^1)^T - a^2(a^2)^T + \langle a^2, a^1\rangle a^2(a^1)^T.$$

5.10 *Let C be a convex set and $f : C \subseteq R^J \to (-\infty, \infty]$. Prove that $f(x)$ is a convex function, according to Definition 5.1, if and only if, for all x and y in C, and for all $0 < \alpha < 1$, we have*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Suppose that the epigraph of f is a convex set. Then

$$\alpha(x, f(x)) + (1-\alpha)(y, f(y)) = (\alpha x + (1-\alpha)y, \alpha f(x) + (1-\alpha)f(y))$$

is a member of the epigraph, which then tells us that

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Conversely, let (x, γ) and (y, δ) be members of the epigraph of f, so that $f(x) \leq \gamma$, and $f(y) \leq \delta$. Let α be in $[0, 1]$. We want to show that

$$\alpha(x, \gamma) + (1-\alpha)(y, \delta) = (\alpha x + (1-\alpha)y, \alpha\gamma + (1-\alpha)\delta)$$

is in the epigraph. But this follows from the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \leq \alpha\gamma + (1-\alpha)\delta.$$

5.11 *Given a point s in a convex set C, where are the points x for which $s = P_C x$?*

Suppose that $s = P_C x$. From the inequality

$$\langle x - s, c - s \rangle \leq 0,$$

for all c in C, it follows that $x - s$ is in $N_C(s)$. So $s = P_C x$ if and only if the vector $x - s$ is in $N_C(s)$.

5.12 *Let C be a closed non-empty convex set in R^J, x a vector not in C, and $d > 0$ the distance from x to C. Let*

$$\sigma_C(a) = \sup_{c \in C} \langle a, c \rangle \,,$$

the support function of C. *Show that*

$$d = m = \max_{||a|| \leq 1} \{\langle a, x \rangle - \sigma_C(a)\}.$$

Hints: Consider the unit vector $\frac{1}{d}(x - P_C x)$, and use Cauchy's Inequality and Proposition 5.2.

Let $a^0 = \frac{1}{d}(x - P_C x)$. First, show that

$$\sigma_C(a^0) = \langle a^0, P_C x \rangle.$$

Then

$$d = \langle a^0, x - P_C x \rangle = \langle a^0, x \rangle - \langle a^0, PCx \rangle = \langle a^0, x \rangle - \sigma_C(a^0) \leq m;$$

therefore, $d \leq m$.

From Cauchy's Inequality, we know that

$$\langle a, x - P_C x \rangle \leq \|x - P_C x\| \|a\| = d,$$

for any vector a with norm one. Therefore, for any a with norm one, we have

$$d \geq \langle a, x \rangle - \langle a, P_C x \rangle \geq \langle a, x \rangle - \sigma_C(a).$$

Therefore, we can conclude that $d \geq m$, and so $d = m$.

5.13 (Rådström Cancellation)

- *(a) Show that, for any subset S of R^J, we have $2S \subseteq S + S$, and $2S = S + S$ if S is convex.*

- *(b) Find three finite subsets of R, say A, B, and C, with A not contained in B, but with the property that $A + C \subseteq B + C$. Hint: try to find an example where the set C is $C = \{-1, 0, 1\}$.*

- *(c) Show that, if A and B are convex, B is closed, and C is bounded, then $A + C \subseteq B + C$ implies that $A \subseteq B$. Hint: Note that, under these assumptions, $2A + C = A + (A + C) \subseteq 2B + C$.*

- (a) If x and y are in S, then $x + y$ is in $S + S$. If S is convex, then $\frac{x+y}{2}$ is also in S, or
$$x + y = 2(\frac{x + y}{2})$$
is in $2S$; therefore $S + S = 2S$.

- (b) Take $A = \{0, 1\}$, $B = \{0, 2\}$ and $C = \{-1, 0, 1\}$. Then we have
$$A + C = \{-1, 0, 1, 2\},$$
and
$$B + C = \{-1, 0, 1, 2, 3\}.$$

- (c) Begin with
$$2A + C = (A + A) + C = A + (A + C) \subseteq A + (B + C) = (A + C) + B$$
$$\subseteq (B + C) + B = (B + B) + C = 2B + C.$$
Continuing in this way, show that
$$nA + C \subseteq nB + C,$$
or
$$A + \frac{1}{n}C \subseteq B + \frac{1}{n}C,$$
for $n = 1, 2, 3, ...$ Now select a point a in A; we show that a is in B.

  Since C is a bounded set, we know that there is a constant $K > 0$ such that $\|c\| \leq K$, for all c in C. For each n there are c^n and d^n in C and b^n in B such that
$$a + \frac{1}{n}c^n = b^n + \frac{1}{n}d^n.$$
Then we have
$$b^n = a + \frac{1}{n}(c^n - d^n). \tag{5.7}$$
Since C is bounded, we know that $\frac{1}{n}(c^n - d^n)$ converges to zero, as $n \to \infty$. Then, taking limits on both sides of Equation (5.7), we find that $\{b^n\} \to a$, which must be in B, since B is closed.

Chapter 6

Linear Programming

6.1 Needed References

Theorem 6.1 *Let x and y be feasible vectors. Then*

$$z = c^T x \geq b^T y = w. \tag{6.1}$$

Corollary 6.1 *If z is not bounded below, then there are no feasible y.*

Corollary 6.2 *If x and y are both feasible, and $z = w$, then both x and y are optimal for their respective problems.*

Lemma 6.1 *Let $W = \{w^1, ..., w^N\}$ be a spanning set for a subspace S in R^I, and $V = \{v^1, ..., v^M\}$ a linearly independent subset of S. Then $M \leq N$.*

6.2 Exercises

6.1 *Prove Theorem 6.1 and its corollaries.*

Since y is feasible, we know that $A^T y \leq c$, and since x is feasible, we know that $x \geq 0$ and $Ax = b$. Therefore, we have

$$w = b^T y = (Ax)^T y = x^T A^T y \leq x^T c = c^T x = z.$$

This tells us that, if there are feasible x and y, then $z = c^T x$ is bounded below, as we run over all feasible x, by any w. Consequently, if z is not bounded, there cannot be any w, so there can be no feasible y.

If both x and y are feasible and $z = c^T x = b^T y = w$, then we cannot decrease z by replacing x, nor can we increase w by replacing y. Therefore, x and y are the best we can do.

6.2 *Let $W = \{w^1, ..., w^N\}$ be a spanning set for a subspace S in R^I, and $V = \{v^1, ..., v^M\}$ a linearly independent subset of S. Let A be the matrix whose columns are the v^m, B the matrix whose columns are the w^n. Show that there is an N by M matrix C such that $A = BC$. Prove Lemma 6.1 by showing that, if $M > N$, then there is a non-zero vector x with $Cx = Ax = 0$.*

If C is any M by N matrix with $M > N$, then, using Gauss elimination, we can show that there must be non-zero solutions of $Cx = 0$. Now, since W is a spanning set, each column of A is a linear combination of the columns of B, which means that we can find some C such that $A = BC$. If there is a non-zero x for which $Cx = 0$, then $Ax = 0$ also. But the columns of A are linearly independent, which tells us that $Ax = 0$ has only the zero solution. We conclude that $M \leq N$.

6.3 *Show that when the simplex method has reached the optimal solution for the primal problem PS, the vector y with $y^T = c_B^T B^{-1}$ becomes a feasible vector for the dual problem and is therefore the optimal solution for DS. Hint: Clearly, we have*

$$z = c^T x = c_B^T B^{-1} b = y^T b = w,$$

so we need only show that $A^T y \leq c$.

We know that the simplex algorithm halts when the vector

$$r^T = (c_N^T - c_B^T B^{-1} N) = c_N^T - y^T N$$

has only non-negative entries, or when

$$c_N \geq N^T y.$$

Then we have

$$A^T y = \begin{bmatrix} B^T \\ N^T \end{bmatrix} y = \begin{bmatrix} B^T y \\ N^T y \end{bmatrix} \leq \begin{bmatrix} B^T (B^T)^{-1} c_B \\ c_N \end{bmatrix} = c,$$

so y is feasible, and therefore optimal.

Chapter 7

Matrix Games and Optimization

7.1 Some Exercises

7.1 *Show that the vectors $\hat{p} = \frac{1}{\mu}\hat{x}$ and $\hat{q} = \frac{1}{\mu}\hat{y}$ are probability vectors and are optimal randomized strategies for the matrix game.*

Since

$$\hat{z} = c^T \hat{x} = \mu = b^T \hat{y} = \hat{w},$$

it follows that $c^T \hat{p} = b^T \hat{q} = 1$. We also have

$$\hat{x}^T A \hat{y} \leq \hat{x}^T c = \mu,$$

and

$$\hat{x}^T A \hat{y} = (A^T \hat{x})^T \hat{y} \geq b^T \hat{y} = \mu,$$

so that

$$\hat{x}^T A \hat{y} = \mu,$$

and

$$\hat{p}^T A \hat{q} = \frac{1}{\mu}.$$

For any probabilities p and q we have

$$p^T A \hat{q} = \frac{1}{\mu} p^T A \hat{y} \leq \frac{1}{\mu} p^T c = \frac{1}{\mu},$$

and

$$\hat{p}^T A q = (A^T \hat{p})^T q = \frac{1}{\mu}(A^T \hat{x})^T q \geq \frac{1}{\mu} b^T q = \frac{1}{\mu}.$$

7.2 *Given an arbitrary I by J matrix A, there is $\alpha > 0$ so that the matrix B with entries $B_{ij} = A_{ij} + \alpha$ has only positive entries. Show that any optimal randomized probability vectors for the game with pay-off matrix B are also optimal for the game with pay-off matrix A.*

This one is easy.

Chapter 8

Differentiation

8.1 Exercises

8.1 *Let Q be a real, positive-definite symmetric matrix. Define the Q-inner product on R^J to be*

$$\langle x, y\rangle_Q = x^T Q y = \langle x, Qy\rangle,$$

and the Q-norm to be

$$||x||_Q = \sqrt{\langle x, x\rangle_Q}.$$

Show that, if $\nabla f(a)$ is the Fréchet derivative of $f(x)$ at $x = a$, for the usual Euclidean norm, then $Q^{-1}\nabla f(a)$ is the Fréchet derivative of $f(x)$ at $x = a$, for the Q-norm. Hint: use the inequality

$$\sqrt{\lambda_J}||h||_2 \leq ||h||_Q \leq \sqrt{\lambda_1}||h||_2,$$

where λ_1 and λ_J denote the greatest and smallest eigenvalues of Q, respectively.

8.2 *For (x, y) not equal to $(0, 0)$, let*

$$f(x, y) = \frac{x^a y^b}{x^p + y^q},$$

with $f(0, 0) = 0$. In each of the five cases below, determine if the function is continuous, Gâteaux, Fréchet or continuously differentiable at $(0, 0)$.

- *1) $a = 2$, $b = 3$, $p = 2$, and $q = 4$;*
- *2) $a = 1$, $b = 3$, $p = 2$, and $q = 4$;*
- *3) $a = 2$, $b = 4$, $p = 4$, and $q = 8$;*

- *4)* $a = 1$, $b = 2$, $p = 2$, *and* $q = 2$*;*
- *5)* $a = 1$, $b = 2$, $p = 2$, *and* $q = 4$.

Chapter 9

Convex Functions

Proposition 9.1 *The following are equivalent:*
1) the epi-graph of $g(x)$ is convex;
2) for all points $a < x < b$

$$g(x) \leq \frac{g(b) - g(a)}{b - a}(x - a) + g(a); \tag{9.1}$$

3) for all points $a < x < b$

$$g(x) \leq \frac{g(b) - g(a)}{b - a}(x - b) + g(b); \tag{9.2}$$

4) for all points a and b in R and for all α in the interval $(0, 1)$

$$g((1 - \alpha)a + \alpha b) \leq (1 - \alpha)g(a) + \alpha g(b). \tag{9.3}$$

Lemma 9.1 *A firmly non-expansive operator on R^J is non-expansive.*

9.1 Exercises

9.1 *Prove Proposition 9.1.*

The key idea here is to use $\alpha = \frac{x-a}{b-a}$, so that $x = (1 - \alpha)a + \alpha b$.

9.2 *Prove Lemma 9.1.*

From the definition, F is firmly non-expansive if

$$\langle F(x) - F(y), x - y\rangle \geq \|x - y\|_2^2. \tag{9.4}$$

Now use the Cauchy Inequality on the left side.

9.3 *Show that, if $\hat{x}$ minimizes the function $g(x)$ over all x in R^J, then $x = 0$ is in the sub-differential $\partial g(\hat{x})$.*

This is easy.

9.4 *If $f(x)$ and $g(x)$ are convex functions on R^J, is $f(x) + g(x)$ convex? Is $f(x)g(x)$ convex?*

It is easy to show that the sum of two convex functions is again convex. The product of two is, however, not always convex; take $f(x) = -1$ and $g(x) = x^2$, for example.

9.5 *Let $\iota_C(x)$ be the indicator function of the closed convex set C, that is,*

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Show that the sub-differential of the function ι_C at a point c in C is the normal cone to C at the point c, that is, $\partial \iota_C(c) = N_C(c)$, for all c in C.

This follows immediately from the definitions.

9.6 *Let $g(t)$ be a strictly convex function for $t > 0$. For $x > 0$ and $y > 0$, define the function*

$$f(x, y) = xg(\frac{y}{x}).$$

Use induction to prove that

$$\sum_{n=1}^{N} f(x_n, y_n) \geq f(x_+, y_+),$$

for any positive numbers x_n and y_n, where $x_+ = \sum_{n=1}^{N} x_n$. Also show that equality obtains if and only if the finite sequences $\{x_n\}$ and $\{y_n\}$ are proportional.

We show this for the case of $N = 2$; the more general case is similar. We need to show that

$$f(x_1, y_1) + f(x_2, y_2) \geq f(x_1 + x_2, y_1 + y_2).$$

Write

$$f(x_1, y_1) + f(x_2, y_2) = x_1 g(\frac{y_1}{x_1}) + x_2 g(\frac{y_2}{x_2}) = (x_1 + x_2)\Big(\frac{x_1}{x_1 + x_2} g(\frac{y_1}{x_1}) + \frac{x_2}{x_1 + x_2} g(\frac{y_2}{x_2})\Big)$$

$$\geq (x_1 + x_2) g(\frac{x_1}{x_1 + x_2}\frac{y_1}{x_1} + \frac{x_2}{x_1 + x_2}\frac{y_2}{x_2}) = x_+ g(\frac{y_+}{x_+}) = f(x_+, y_+).$$

9.7 *Use the result in Exercise 9.6 to obtain Cauchy's Inequality. Hint: let* $g(t) = -\sqrt{t}$.

Using the result in Exercise 9.6, and the choice of $g(t) = -\sqrt{t}$, we obtain

$$-\sum_{n=1}^{N} x_n \sqrt{\frac{y_n}{x_n}} \geq -x_+ \sqrt{\frac{y_+}{x_+}}$$

so that

$$\sum_{n=1}^{N} \sqrt{x_N}\sqrt{y_n} \leq \sqrt{x_+}\sqrt{y_+},$$

which is Cauchy's Inequality.

9.8 *Use the result in Exercise 9.6 to obtain* **Milne's Inequality***:*

$$x_+y_+ \geq \Big(\sum_{n=1}^{N}(x_n + y_n)\Big)\Big(\sum_{n=1}^{N} \frac{x_n y_n}{x_n + y_n}\Big).$$

Hint: let $g(t) = -\frac{t}{1+t}$.

This is a direct application of the result in Exercise 9.6.

9.9 *For* $x > 0$ *and* $y > 0$*, let* $f(x,y)$ *be the* **Kullback-Leibler function***,*

$$f(x,y) = KL(x,y) = x\Big(\log\frac{x}{y}\Big) + y - x.$$

Use Exercise 9.6 to show that

$$\sum_{n=1}^{N} KL(x_n, y_n) \geq KL(x_+, y_+).$$

Use $g(t) = -\log t$.

Chapter 10

Fenchel Duality

10.1 Exercises

10.1 *Let A be a real symmetric positive-definite matrix and*

$$f(x) = \frac{1}{2}\langle Ax, x\rangle.$$

Show that

$$f^*(a) = \frac{1}{2}\langle A^{-1}a, a\rangle.$$

Hints: Find $\nabla f(x)$ and use Equation (10.1).

We have

$$f^*(a) = \langle a, (\nabla f)^{-1}(a)\rangle - f((\nabla f)^{-1}(a)). \tag{10.1}$$

Since $\nabla f(x) = Ax$, it follows from Equation (10.1) that

$$f^*(a) = \langle a, A^{-1}a\rangle - \frac{1}{2}\langle A(A^{-1})a, A^{-1}a\rangle = \frac{1}{2}\langle a, A^{-1}a\rangle.$$

Chapter 11

Convex Programming

11.1 Referenced Results

Theorem 11.1 *Let $(\hat{x}, \hat{y})$ be a saddle point for $K(x, y)$. Then $\hat{x}$ solves the primal problem, that is, $\hat{x}$ minimizes $f(x)$, over all x in X, and $\hat{y}$ solves the dual problem, that is, $\hat{y}$ maximizes $g(y)$, over all y in Y. In addition, we have*

$$g(y) \leq K(\hat{x}, \hat{y}) \leq f(x), \tag{11.1}$$

for all x and y, so that the maximum value of $g(y)$ and the minimum value of $f(x)$ are both equal to $K(\hat{x}, \hat{y})$.

Theorem 11.2 *Let x^* be a regular point. If x^* is a local constrained minimizer of $f(x)$, then there is a vector λ^* such that*

- **1)** $\lambda_i^* \geq 0$, *for* $i = 1, ..., K$;
- **2)** $\lambda_i^* g_i(x^*) = 0$, *for* $i = 1, ..., K$;
- **3)** $\nabla f(x^*) + \sum_{i=1}^I \lambda_i^* \nabla g_i(x^*) = 0$.

11.2 Exercises

11.1 *Prove Theorem 11.1.*

We have

$$f(\hat{x}) = \sup_y K(\hat{x}, y) = K(\hat{x}, \hat{y}),$$

and

$$f(x) = \sup_y K(x, y) \geq K(x, \hat{y}) \geq K(\hat{x}, \hat{y}) = f(\hat{x}).$$

From $f(x) \geq f(\hat{x})$ we conclude that $x = \hat{x}$ minimizes $f(x)$. In a similar way, we can show that $y = \hat{y}$ maximizes $g(y)$.

11.2 *Apply the gradient form of the KKT Theorem to minimize the function $f(x,y) = (x+1)^2 + y^2$ over all $x \geq 0$ and $y \geq 0$.*

Setting the x-gradient of the Lagrangian to zero, we obtain the equations

$$2(x+1) - \lambda_1 = 0,$$

and

$$2y - \lambda_2 = 0.$$

Since $x \geq 0$, we cannot have $\lambda_1 = 0$, consequently $g_1(x,y) = -x = 0$, so $x = 0$. We also have that $y = 0$ whether or not $\lambda_2 = 0$. Therefore, the answer is $x = 0$, $y = 0$.

11.3 *Minimize the function*

$$f(x,y) = \sqrt{x^2 + y^2},$$

subject to

$$x + y \leq 0.$$

Show that the function $MP(z)$ is not differentiable at $z = 0$.

Equivalently, we minimize $f(x,y)^2 = x^2 + y^2$, subject to the constraint $g(x,y) \leq z$. If $z \geq 0$, the optimal point is $(0,0)$ and $MP(z) = 0$. If $z < 0$, the optimal point is $x = \frac{z}{2} = y$, and $MP(z) = \frac{-z}{\sqrt{2}}$. The directional derivative of $MP(z)$ at $z = 0$, in the positive direction is zero, while in the negative direction it is $\frac{1}{\sqrt{2}}$, so $MP(z)$ is not differentiable at $z = 0$.

11.4 *Minimize the function*

$$f(x,y) = -2x - y,$$

subject to

$$x + y \leq 1,$$

$$0 \leq x \leq 1,$$

and

$$y \geq 0.$$

We write $g_1(x,y) = x + y - 1$, $g_2(x,y) = -x \leq 0$, $g_3(x,y) = x - 1 \leq 0$, and $g_4(x,y) = -y \leq 0$. Setting the partial derivatives of the Lagrangian to zero, we get

$$0 = -2 + \lambda_1 + \lambda_2 - \lambda_3,$$

and

$$0 = -1 + \lambda_1 - \lambda_4.$$

Since $\lambda_i \geq 0$ for each i, it follows that $\lambda_1 \neq 0$, so that $0 = g_1(x, y) = x + y - 1$, or $x + y = 1$.

If $\lambda_4 = 0$, then $\lambda_1 = 1$ and $\lambda_2 - \lambda_3 = 1$. Therefore, we cannot have $\lambda_2 = 0$, which tells us that $g_2(x, y) = 0$, or $x = 1$, in addition to $y = 0$.

If $\lambda_4 > 0$, then $y = 0$, which we already know, and so $x = 1$ again. In any case, then answer must be $x = 1$ and $y = 0$, so that the minimum is $f(1, 0) = -2$.

11.5 *Apply the theory of convex programming to the primal Quadratic Programming Problem (QP), which is to minimize the function*

$$f(x) = \frac{1}{2}x^T Qx,$$

subject to

$$a^T x \leq c,$$

where $a \neq 0$ is in R^J, $c < 0$ is real, and Q is symmetric, and positive-definite.

With $g(x) = a^T x - c \leq 0$, we set the x-gradient of $L(x, \lambda)$ equal to zero, obtaining

$$Qx^* + \lambda a = 0,$$

or

$$x^* = -\lambda Q^{-1} a.$$

So, for any $\lambda \geq 0$, $\nabla_x L(x, \lambda) = 0$ has a solution, so all $\lambda \geq 0$ are feasible for the dual problem. We can then write

$$L(x^*, \lambda) = \frac{\lambda^2}{2} a^T Q^{-1} a - \lambda^2 a^T Q^{-1} a - \lambda c = -\Big(\frac{\lambda^2}{2} a^T Q^{-1} a + \lambda c\Big).$$

Now we maximize $L(x^*, \lambda)$ over $\lambda \geq 0$.

We get

$$0 = -\lambda a^T Q^{-1} a - c,$$

so that

$$\lambda^* = -\frac{c}{a^T Q^{-1} a},$$

and the optimal x^* is

$$x^* = \frac{cQ^{-1} a}{a^T Q^{-1} a}.$$

11.6 *Use Theorem 11.2 to prove that any real N by N symmetric matrix has N mutually orthonormal eigenvectors.*

Let Q be symmetric. First, we minimize $f(x) = \frac{1}{2}x^TQx$ subject to $g(x) = 1 - x^Tx = 0$. The feasible set is closed and bounded, and $f(x)$ is continuous, so there must be a minimum. From the KKT Theorem we have

$$Qx - \lambda x = 0,$$

so that $Qx = \lambda x$ and x is an eigenvector of Q; call $x = x^N$, and $\lambda = \gamma_N$. Note that the optimal value of $f(x)$ is $\frac{\gamma_N}{2}$.

Next, minimize $f(x)$, subject to $g_1(x) = 1 - x^Tx = 0$ and $x^Tx^N = 0$. The Lagrangian is

$$L(x, \lambda) = \frac{1}{2}x^TQx + \lambda_1(1 - x^Tx) + \lambda_2 x^Tx^N.$$

Setting the x-gradient of $L(x, \lambda)$ to zero, we get

$$0 = Qx - \lambda_1 x + \lambda_2 x^N.$$

Therefore,

$$0 = x^TQx - \lambda_1 x^Tx + \lambda_2 x^Tx^N = x^TQx - \lambda_1,$$

or

$$\lambda_1 = x^TQx.$$

We also have

$$0 = (x^N)^TQx - \lambda_1(x^N)^Tx + \lambda_2(x^N)^Tx^N,$$

and

$$(x^N)^TQ = \gamma_N(x^N)^T,$$

so that

$$0 = \gamma_N(x^N)^Tx - \lambda_1(x^N)^Tx + \lambda_2 = \lambda_2,$$

so $\lambda_2 = 0$. Therefore, we have

$$Qx = \lambda_1 x,$$

so x is another eigenvector of Q, with associated eigenvalue λ_1; we write $x = x^{N-1}$, and $\lambda_1 = \gamma_{N-1}$. Note that the optimal value of $f(x)$ is now $\frac{\gamma_{N-1}}{2}$. Since the second optimization problem has more constraints than the first one, we must conclude that $\gamma_{N-1} \geq \gamma_N$.

We continue in this manner, each time including one more orthogonality constraint, which thereby increases the optimal value. When we have performed N optimizations, and are about to perform one more, we find that the constraint conditions now require x to be orthogonal to each of the previously calculated x^n. But these vectors are linearly independent in R^N, and so the only vector orthogonal to all of them is zero, and we are finished.

Chapter 12

Iterative Optimization

12.1 Referenced Results

Lemma 12.1 *Suppose that x^{k+1} is chosen using the optimal value of γ_k, as described by Equation (12.1),*

$$g(x^k - \gamma_k \nabla g(x^k)) \leq g(x^k - \gamma \nabla g(x^k)). \tag{12.1}$$

Then

$$\langle \nabla g(x^{k+1}), \nabla g(x^k) \rangle = 0. \tag{12.2}$$

12.2 Exercises

12.1 *Prove Lemma 12.1.*

Use the Chain Rule to calculate the derivative of the function $f(\gamma)$ given by

$$f(\gamma) = g(x^k - \gamma \nabla g(x^k)),$$

and then set this derivative equal to zero.

12.2 *Apply the Newton-Raphson method to obtain an iterative procedure for finding $\sqrt{a}$, for any positive a. For which x^0 does the method converge? There are two answers, of course; how does the choice of x^0 determine which square root becomes the limit?*

This is best done as a computer exercise.

12.3 *Apply the Newton-Raphson method to obtain an iterative procedure for finding $a^{1/3}$, for any real a. For which x^0 does the method converge?*

Another computer exercise.

12.4 *Extend the Newton-Raphson method to complex variables. Redo the previous exercises for the case of complex a. For the complex case, a has two square roots and three cube roots. How does the choice of x^0 affect the limit? Warning: The case of the cube root is not as simple as it may appear, and has a close connection to fractals and chaos.*

Consult Schroeder's book before trying this exercise.

12.5 (The Sherman-Morrison-Woodbury Identity) *Let A be an invertible matrix. Show that, if $\omega = 1 + v^T A^{-1} u \neq 0$, then $A + uv^T$ is invertible and*

$$(A + uv^T)^{-1} = A^{-1} - \frac{1}{\omega} A^{-1} uv^T A^{-1}. \tag{12.3}$$

This is easy.

12.6 *Use the reduced Newton-Raphson method to minimize the function $\frac{1}{2}x^T Qx$, subject to $Ax = b$, where*

$$Q = \begin{bmatrix} 0 & -13 & -6 & -3 \\ -13 & 23 & -9 & 3 \\ -6 & -9 & -12 & 1 \\ -3 & 3 & 1 & -1 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & -1 \end{bmatrix},$$

and

$$b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Start with

$$x^0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We begin by finding a basis for the null space of A, which we do by using Gauss elimination to solve $Ax = 0$. We find that $(1, -4, 1, 0)^T$ and $(-2, 3, 0, 1)^T$ do the job, so the matrix Z is

$$Z = \begin{bmatrix} 1 & -2 \\ -4 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix Z^TQ is then

$$Z^TQ = \begin{bmatrix} 46 & -114 & 18 & -14 \\ -42 & 98 & -14 & 14 \end{bmatrix}$$

and

$$Z^TQx^0 = \begin{bmatrix} -68 \\ 56 \end{bmatrix}.$$

We have

$$Z^TQZ = \begin{bmatrix} 520 & -448 \\ -448 & 392 \end{bmatrix},$$

so that

$$(Z^TQZ)^{-1} = \frac{1}{3136}\begin{bmatrix} 392 & 448 \\ 448 & 520 \end{bmatrix}.$$

One step of the reduced Newton-Raphson algorithm, beginning with $v^0 = 0$, gives

$$v^1 = -(Z^TQZ)^{-1}Z^TQx^0 = \begin{bmatrix} 0.5 \\ 0.4286 \end{bmatrix},$$

so that

$$x^1 = x^0 + Zv^1 = \begin{bmatrix} 0.6428 \\ 0.2858 \\ 0.5 \\ 0.4286 \end{bmatrix}.$$

When we check, we find that $Z^TQx^1 = 0$, so we are finished.

12.7 *Use the reduced steepest descent method with an exact line search to solve the problem in the previous exercise.*

Do as a computer problem.

Chapter 13

Modified-Gradient Algorithms

There are no exercises in this chapter.

Chapter 14

Quadratic Programming

There are no exercises in this chapter.

Chapter 15

Solving Systems of Linear Equations

15.1 Regularizing the ART

In our first method we use ART to solve the system of equations given in matrix form by

$$[\,A^T \quad \epsilon I\,]\begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{15.1}$$

We begin with $u^0 = b$ and $v^0 = 0$. Then, the lower component of the limit vector is $v^\infty = -\epsilon \hat{x}_\epsilon$.

The method of Eggermont *et al.* is similar. In their method we use ART to solve the system of equations given in matrix form by

$$[\,A \quad \epsilon I\,]\begin{bmatrix} x \\ v \end{bmatrix} = b. \tag{15.2}$$

We begin at $x^0 = 0$ and $v^0 = 0$. Then, the limit vector has for its upper component $x^\infty = \hat{x}_\epsilon$, and $\epsilon v^\infty = b - A\hat{x}_\epsilon$.

15.2 Exercises

15.1 *Show that the two algorithms associated with Equations (15.1) and (15.2), respectively, do actually perform as claimed.*

We begin by recalling that, in the under-determined case, the minimum two-norm solution of a system of equations $Ax = b$ has the form $x = A^T z$, for some z, so that the minimum two-norm solution is $x = A^T(AA^T)^{-1}b$, provided that the matrix AA^T is invertible, which it usually is in the under-determined case.

The solution of $Ax = b$ for which $\|x - p\|$ is minimized can be found in a similar way. We let $z = x - p$, so that $x = z + p$ and $Az = b - Ap$. Now we find the minimum two-norm solution of the system $Az = b - Ap$; our final solution is $x = z + p$.

The regularized solution $\hat{x}_\epsilon$ that we seek minimizes the function

$$f(x) = \frac{1}{2}\|Ax - b\|^2 + \epsilon^2\|x\|^2,$$

and therefore can be written explicitly as

$$\hat{x}_\epsilon = (A^TA + \epsilon^2 I)^{-1}A^Tb.$$

For large systems, it is too expensive and time-consuming to calculate $\hat{x}_\epsilon$ this way; therefore, we seek iterative methods, in particular, ones that do not require the calculation of the matrix A^TA.

The first of the two methods offered in the chapter has us solve the system

$$[\,A^T \quad \epsilon I\,]\begin{bmatrix} u \\ v \end{bmatrix} = 0. \tag{15.3}$$

When we use the ART algorithm, we find the solution closest to where we began the iteration. We begin at

$$\begin{bmatrix} u^0 \\ v^0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

so we are finding the solution of Equation (15.1) for which

$$\|u - b\|^2 + \|v\|^2$$

is minimized. From our previous discussion, we see that we need to find the solution of

$$[\,A^T \quad \epsilon I\,]\begin{bmatrix} s \\ t \end{bmatrix} = -A^Tb$$

for which

$$\|s\|^2 + \|t\|^2$$

is minimized. This minimum two-norm solution must have the form

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} A \\ \epsilon I \end{bmatrix} z = \begin{bmatrix} Az \\ \epsilon z \end{bmatrix}.$$

This tells us that

$$(A^TA + \epsilon^2 I)z = -A^Tb,$$

or that $-z = \hat{x}_\epsilon$. Since the lower part of the minimum two-norm solution is $t = \epsilon z$, the assertion concerning this algorithm is established.

The second method has us solve the system

$$[\,A \quad \epsilon I\,] \begin{bmatrix} x \\ v \end{bmatrix} = b, \tag{15.4}$$

using the ART algorithm and beginning at

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so we are seeking a minimum two-norm solution of the system in Equation (15.2). We know that this minimum two-norm solution must have the form

$$\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} A^T \\ \epsilon I \end{bmatrix} z = \begin{bmatrix} A^T z \\ \epsilon z \end{bmatrix}.$$

Therefore,

$$(AA^T + \epsilon^2 I)z = b,$$

and

$$A^T(AA^T + \epsilon^2 I)z = (A^T A + \epsilon^2 I)A^T z = (A^T A + \epsilon^2 I)x = b.$$

Consequently, $x = \hat{x}_\epsilon$, and this x is the upper part of the limit vector of the ART iteration.

Chapter 16

Conjugate-Direction Methods

16.1 Proofs of Lemmas

Lemma 16.1 *When x^k is constructed using the optimal α, we have*

$$\nabla f(x^k) \cdot d^k = 0. \tag{16.1}$$

Proof: Differentiate the function $f(x^{k-1}+\alpha d^k)$ with respect to the variable α, and then set it to zero. According to the Chain Rule, we have

$$\nabla f(x^k) \cdot d^k = 0.$$

∎

Lemma 16.2 *The optimal α_k is*

$$\alpha_k = \frac{r^k \cdot d^k}{d^k \cdot Qd^k}, \tag{16.2}$$

where $r^k = c - Qx^{k-1}$.

Proof: We have

$$\nabla f(x^k) = Qx^k - c = Q(x^{k-1} + \alpha_k d^k) - c = Qx^{k-1} - c + \alpha_k Qd^k,$$

so that

$$0 = \nabla f(x^k) \cdot d^k = -r^k \cdot d^k + \alpha d^k \cdot Qd^k.$$

∎

Lemma 16.3 *Let $||x||_Q^2 = x \cdot Qx$ denote the square of the Q-norm of x. Then*

$$||\hat{x} - x^{k-1}||_Q^2 - ||\hat{x} - x^k||_Q^2 = (r^k \cdot d^k)^2 / d^k \cdot Qd^k \geq 0$$

for any direction vectors d^k.

Proof: We use $c = Q\hat{x}$. Then we have

$$(\hat{x} - x^k) \cdot Q(\hat{x} - x^k) = (\hat{x} - x^{k-1}) \cdot Q(\hat{x} - x^{k-1}) - 2\alpha_k d^k \cdot Q(\hat{x} - x^{k-1}) + \alpha_k^2 d^k \cdot Qd^k,$$

so that

$$\|\hat{x} - x^{k-1}\|_Q^2 - \|\hat{x} - x^k\|_Q^2 = 2\alpha_k d^k \cdot (c - Qx^{k-1}) - \alpha_k^2 d^k \cdot Qd^k.$$

Now use $r^k = c - Qx^{k-1}$ and the value of α_k. ∎

Lemma 16.4 *A conjugate set that does not contain zero is linearly independent. If $p^n \neq 0$ for $n = 1, ..., J$, then the least-squares vector $\hat{x}$ can be written as*

$$\hat{x} = a_1 p^1 + ... + a_J p^J,$$

with $a_j = c \cdot p^j / p^j \cdot Qp^j$ for each j.

Proof: Suppose that we have

$$0 = c_1 p^1 + ... + c_n p^n,$$

for some constants $c_1, ..., c_n$. Then, for each $m = 1, ..., n$ we have

$$0 = c_1 p^m \cdot Qp^1 + ... + c_n p^m \cdot Qp^n = c_m p^m \cdot Qp^m,$$

from which it follows that $c_m = 0$ or $p^m = 0$.

Now suppose that the set $\{p^1, ..., p^J\}$ is a conjugate basis for R^J. Then we can write

$$\hat{x} = a_1 p^1 + ... + a_J p^J,$$

for some a_j. Then for each m we have

$$p^m \cdot c = p^m \cdot Q\hat{x} = a_m p^m \cdot Qp^m,$$

so that

$$a_m = \frac{p^m \cdot c}{p^m \cdot Qp^m}.$$

∎

Lemma 16.5 *Whenever $p^{n+1} = 0$, we also have $r^{n+1} = 0$, in which case we have $c = Qx^n$, so that x^n is the least-squares solution.*

Proof: If $p^{n+1} = 0$, then r^{n+1} is a multiple of p^n. But, r^{n+1} is orthogonal to p^n, so $r^{n+1} = 0$. ∎

Theorem 16.1 *For $n = 1, 2, ..., J$ and $j = 1, ..., n-1$ we have*

- **a)** $r^n \cdot r^j = 0$*;*
- **b)** $r^n \cdot p^j = 0$*; and*
- **c)** $p^n \cdot Qp^j = 0$.

16.2 Exercises

16.1 *There are several lemmas in this chapter whose proofs are only sketched. Complete the proofs of these lemma.*

The proof of Theorem 16.1 uses induction on the number n. Throughout the following exercises assume that the statements in the theorem hold for some $n < J$. We prove that they hold also for $n + 1$.

16.2 *Use the fact that*

$$r^{j+1} = r^j - \alpha_j Qp^j,$$

to show that Qp^j is in the span of the vectors r^j and r^{j+1}

We have

$$r^{j+1} = c - Qx^j = c - Qx^{j-1} - \alpha_j Qp^j = r^j - \alpha_j Qp^j,$$

so that

$$r^{j+1} - r^j = -\alpha_j Qp^j.$$

16.3 *Show that $r^{n+1} \cdot r^n = 0$. Hints: establish that*

$$\alpha_n = \frac{r^n \cdot r^n}{p^n \cdot Qp^n},$$

by showing that

$$r^n \cdot p^n = r^n \cdot r^n,$$

and then use the induction hypothesis to show that

$$r^n \cdot Qp^n = p^n \cdot p^n.$$

We know that

$$r^n = p^n + \beta_{n-1}p^{n-1},$$

where

$$b_{n-1} = \frac{r^n \cdot Qp^{n-1}}{p^{n-1} \cdot Qp^{n-1}}.$$

Since $r^n \cdot p^{n-1} = 0$, it follows that

$$r^n \cdot p^n = r^n \cdot r^n.$$

Therefore, we have

$$\alpha_n = \frac{r^n \cdot r^n}{p^n \cdot Qp^n}.$$

From the induction hypothesis, we have

$$(r^n - p^n) \cdot Qp^n = \beta_{n-1}p^{n-1} \cdot Qp^n = 0,$$

so that

$$r^n \cdot Qp^n = p^n \cdot Qp^n.$$

Using

$$r^{n+1} = r^n - \alpha_n Qp^n,$$

we find that

$$r^{n+1} \cdot r^n = r^n \cdot r^n - \alpha_n r^n \cdot Qp^n = 0.$$

16.4 *Show that $r^{n+1} \cdot r^j = 0$, for $j = 1, ..., n-1$. Hints: 1. show that r^{n+1} is in the span of the vectors $\{r^{j+1}, Qp^{j+1}, Qp^{j+2}, ..., Qp^n\}$; and 2. show that r^j is in the span of $\{p^j, p^{j-1}\}$. Then use the induction hypothesis.*

We begin with

$$r^{n+1} = r^n - \alpha_n Qp^n = r^{n-1} - \alpha_{n-1}Qp^{n-1} - \alpha_n Qp^n.$$

We then use

$$r^{n-1} = r^{n-2} - \alpha_{n-2}p^{n-2},$$

and so on, to get that r^{n+1} is in the span of the vectors $\{r^{j+1}, Qp^{j+1}, Qp^{j+2}, ..., Qp^n\}$. We then use

$$r^j \cdot r^{j+1} = 0,$$

and

$$r^j = p^j + \beta_{j-1}p^{j-1},$$

along with the induction hypothesis, to get

$$r^j \cdot Qp^m = 0,$$

for $m = j+1, ..., n$.

16.5 *Show that* $r^{n+1} \cdot p^j = 0$, *for* $j = 1, ..., n$. *Hint: show that* p^j *is in the span of the vectors* $\{r^j, r^{j-1}, ..., r^1\}$, *and then use the previous two exercises.*

Write
$$p^j = r^j - \beta_{j-1} p^{j-1}$$
and repeat this for p^{j-1}, p^{j-2}, and so on, and use $p^1 = r^1$ to show that p^j is in the span of $\{r^j, r^{j-1}, ..., r^1\}$. Then use the previous exercises.

16.6 *Show that* $p^{n+1} \cdot Qp^j = 0$, *for* $j = 1, ..., n-1$. *Hint: use*
$$Qp^j = \alpha_j^{-1}(r^j - r^{j+1}).$$

We have
$$p^{n+1} \cdot Qp^j = r^{n+1} \cdot Qp^j = r^{n+1} \cdot (\alpha_j^{-1}(r^j - r^{j-1})) = 0.$$

The final step in the proof is contained in the following exercise.

16.7 *Show that* $p^{n+1} \cdot Qp^n = 0$. *Hints: write*
$$p^{n+1} = r^{n+1} - \beta_n p^n,$$
where
$$\beta_n = -\frac{r^{n+1} \cdot Qp^n}{p^n \cdot Qp^n}.$$

We have
$$p^{n+1} \cdot Qp^n = r^{n+1} \cdot Qp^n - \beta_n p^n \cdot Qp^n = 0,$$
from the definition of β_n.

Chapter 17

Sequential Unconstrained Minimization Algorithms

Lemma 17.1 *For any non-negative vectors x and z, with $z_+ = \sum_{j=1}^{J} z_j > 0$, we have*

$$KL(x, z) = KL(x_+, z_+) + KL(x, \frac{x_+}{z_+} z). \tag{17.1}$$

17.1 Exercises

17.1 *Prove Lemma 17.1.*

This is easy.

17.2 *Use the logarithmic barrier method to minimize the function*

$$f(x, y) = x - 2y,$$

subject to the constraints

$$1 + x - y^2 \geq 0,$$

and

$$y \geq 0.$$

For $k > 0$, we minimize

$$x - 2y - k \log(1 + x - y^2) - k \log(y).$$

Setting the gradient to zero, we get

$$k = 1 + x - y^2,$$

and

$$2 = k\Big(\frac{2y}{1+x-y^2} - \frac{1}{y}\Big).$$

Therefore,

$$2y^2 - 2y - k = 0,$$

so that

$$y = \frac{1}{2} + \frac{1}{2}\sqrt{1+2k}.$$

As $k \to 0$, we have $y \to 1$ and $x \to 0$, so the optimal solution is $x = 0$ and $y = 1$, which can be checked using the KKT Theorem.

17.3 *Use the quadratic-loss penalty method to minimize the function*

$$f(x, y) = -xy,$$

subject to the equality constraint

$$x + 2y - 4 = 0.$$

For each $k > 0$ we minimize the function

$$-xy + k(x + 2y - 4)^2.$$

Setting the gradient equal to zero, we obtain

$$x = 4k(x + 2y - 4),$$

$$y = 2k(x + 2y - 4),$$

and so $x = 2y$ and

$$y = \frac{2x-8}{-4+\frac{1}{k}}.$$

Solving for x, we get

$$x = \frac{16}{8-\frac{1}{k}}$$

and

$$y = \frac{8}{8-\frac{1}{k}}.$$

As $k \to \infty$, x approaches 2 and y approaches 1. We can check this result by substituting $x = 4 - 2y$ into $f(x, y)$ and minimizing it as a function of y alone.

Chapter 18

Likelihood Maximization

There are no exercises in this chapter.

Chapter 19

Operators

19.1 Referenced Results

Suppose that B is a diagonalizable matrix, that is, there is a basis for R^J consisting of eigenvectors of B. Let $\{u^1, ..., u^J\}$ be such a basis, and let $Bu^j = \lambda_j u^j$, for each $j = 1, ..., J$. For each x in R^J, there are unique coefficients a_j so that

$$x = \sum_{j=1}^{J} a_j u^j. \tag{19.1}$$

Then let

$$||x|| = \sum_{j=1}^{J} |a_j|. \tag{19.2}$$

Lemma 19.1 *The expression $||\cdot||$ in Equation (19.2) defines a norm on R^J. If $\rho(B) < 1$, then the affine operator T is sc, with respect to this norm.*

Lemma 19.2 *Let T be an arbitrary operator T on R^J and $G = I - T$. Then*

$$||x - y||_2^2 - ||Tx - Ty||_2^2 = 2(\langle Gx - Gy, x - y\rangle) - ||Gx - Gy||_2^2. \tag{19.3}$$

Lemma 19.3 *Let T be an arbitrary operator T on R^J and $G = I - T$. Then*

$$\langle Tx - Ty, x - y\rangle - ||Tx - Ty||_2^2 =$$

$$\langle Gx - Gy, x - y\rangle - ||Gx - Gy||_2^2. \tag{19.4}$$

Lemma 19.4 *An operator T is ne if and only if its complement $G = I - T$ is $\frac{1}{2}$-ism, and T is fne if and only if G is 1-ism, and if and only if G is fne. Also, T is ne if and only if $F = (I + T)/2$ is fne. If G is ν-ism and $\gamma > 0$ then the operator γG is $\frac{\nu}{\gamma}$-ism.*

Proposition 19.1 *An operator F is firmly non-expansive if and only if $F = \frac{1}{2}(I + N)$, for some non-expansive operator N.*

Proposition 19.2 *Let T be an affine linear operator whose linear part B is diagonalizable, and $|\lambda| < 1$ for all eigenvalues λ of B that are not equal to one. Then the operator T is pc, with respect to the norm given by Equation (19.2).*

19.2 Exercises

19.1 *Show that a strict contraction can have at most one fixed point.*

Suppose that T is strict contraction, with

$$\|Tx - Ty\| \leq r\|x - y\|,$$

for some $r \in (0, 1)$ and for all x and y. If $Tx = x$ and $Ty = y$, then

$$\|x - y\| = \|Tx - Ty\| \leq r\|x - y\|,$$

which implies that $\|x - y\| = 0$.

19.2 *Let T be sc. Show that the sequence $\{T^k x_0\}$ is a Cauchy sequence.*

From

$$||x^k - x^{k+n}|| \leq ||x^k - x^{k+1}|| + ... + ||x^{k+n-1} - x^{k+n}||. \tag{19.5}$$

and

$$||x^{k+m} - x^{k+m+1}|| \leq r^m ||x^k - x^{k+1}|| \tag{19.6}$$

we have

$$||x^k - x^{k+n}|| \leq (1 + r + r^2 + ... + r^{n-1})\|x^k - x^{k+1}\|.$$

From

$$\|x^k - x^{k+1}\| \leq r^k \|x^0 - x^1\|$$

we conclude that, given any $\epsilon > 0$, we can find $k > 0$ so that, for any $n > 0$

$$\|x^k - x^{k+n}\| \leq \epsilon,$$

so that the sequence $\{T^k x^0\}$ is a Cauchy sequence.

Since $\{x^k\}$ is a Cauchy sequence, it has a limit, say $\hat{x}$. Let $e^k = \hat{x} - x^k$. From

$$e^k = T^k \hat{x} - T^k x^0$$

we have

$$\|e^k\| \leq r^k \|\hat{x} - x^0\|,$$

so that $\{e^k\}$ converges to zero, and $\{x^k\}$ converges to $\hat{x}$. Since the sequence $\{x^{k+1}\}$ is just the sequence $\{x^k\}$, but starting at x^1 instead of x^0, the sequence $\{x^{k+1}\}$ also converges to $\hat{x}$. But we have $\{x^{k+1}\} = \{Tx^k\}$, which converges to $T\hat{x}$, by the continuity of T. Therefore, $T\hat{x} = \hat{x}$.

19.3 *Suppose that we want to solve the equation*

$$x = \frac{1}{2}e^{-x}.$$

Let $Tx = \frac{1}{2}e^{-x}$ for x in R. Show that T is a strict contraction, when restricted to non-negative values of x, so that, provided we begin with $x^0 > 0$, the sequence $\{x^k = Tx^{k-1}\}$ converges to the unique solution of the equation.

Let $0 < x < z$. From the Mean Value Theorem we know that, for $t > 0$,

$$e^{-t} = e^{-0} - e^{-s}t,$$

for some s in the interval $(0, t)$. Therefore,

$$1 - e^{-(z-x)} = e^{-c}(z - x),$$

for some c in the interval $(0, z - x)$. Then

$$|Tx - Tz| = \frac{1}{2}e^{-x}(1 - e^{-(z-x)}) \leq \frac{1}{2}e^{-x}e^{-c}(z - x) \leq \frac{1}{2}(z - x).$$

Therefore, T is a strict contraction, with $r = \frac{1}{2}$.

19.4 *Prove Lemma 19.1.*

Showing that it is a norm is easy.

Since $Tx - Ty = Bx - By$, we know that $\|Tx - Ty\| = \|B(x - y)\|$. Suppose that

$$x - y = \sum_{j=1}^{J} c_j u^j.$$

Then

$$B(x - y) = \sum_{j=1}^{J} \lambda_j c_j u^j,$$

and

$$\|B(x-y)\| = \sum_{j=1}^{J} |\lambda_j||c_j| \leq \rho(B) \sum_{j=1}^{J} |c_j| = \rho(B)\|x-y\|.$$

Therefore, T is a strict contraction in this norm.

19.5 *Show that, if the operator T is α-av and $1 > \beta > \alpha$, then T is β-av.*

Clearly, if

$$\langle Gx - Gy, x - y \rangle \geq \frac{1}{2\alpha}||Gx - Gy||_2^2, \tag{19.7}$$

and $1 > \beta > \alpha$, then

$$\langle Gx - Gy, x - y \rangle \geq \frac{1}{2\beta}||Gx - Gy||_2^2.$$

19.6 *Prove Lemma 19.4.*

Clearly, the left side of the equation

$$||x - y||_2^2 - ||Tx - Ty||_2^2 = 2(\langle Gx - Gy, x - y \rangle) - ||Gx - Gy||_2^2$$

is non-negative if and only if the right side is non-negative, which is equivalent to

$$(\langle Gx - Gy, x - y \rangle) \geq \frac{1}{2}||Gx - Gy||_2^2,$$

which means that G is $\frac{1}{2}$-ism. Similarly, the left side of the equation

$$\langle Tx - Ty, x - y \rangle - ||Tx - Ty||_2^2 =$$

$$\langle Gx - Gy, x - y \rangle - ||Gx - Gy||_2^2$$

is non-negative if and only if the right side is, which says that T is fne if and only if G is fne if and only if G is 1-ism.

19.7 *Prove Proposition 19.1.*

Note that $F = \frac{1}{2}(I + N)$ is equivalent to $F = I - \frac{1}{2}G$, for $G = I - N$. Therefore, if N is ne, then G is $\frac{1}{2}$-ism and $\frac{1}{2}G$ is 1-ism, or fne. Therefore, N is ne if and only if F is the complement of a 1-ism operator, and so is itself a 1-ism operator and fne.

19.8 *Prove Proposition 19.2.*

Suppose that $Ty = y$. We need to show that

$$\|Tx - y\| < \|x - y\|,$$

unless $Tx = x$. Since $Ty = y$, we can write

$$\|Tx - y\| = \|Tx - Ty\| = \|B(x - y)\|.$$

Suppose that

$$x - y = \sum_{j=1}^{J} a_j u^j,$$

and suppose that $Tx \neq x$. Then

$$\|x - y\| = \sum_{j=1}^{J} |a_j|,$$

while

$$\|B(x - y)\| = \sum_{j=1}^{J} |\lambda_j||a_j|.$$

If $\lambda_j = 1$ for all j such that $a_j \neq 0$, then $B(x - y) = x - y$, which is not the case, since $Tx - Ty \neq x - y$. Therefore, at least one $|\lambda_j|$ is less than one, and so

$$\|Tx - y\| < \|x - y\|.$$

19.9 *Show that, if B is a linear av operator, then $|\lambda| < 1$ for all eigenvalues λ of B that are not equal to one.*

We know that B is av if and only if there is $\alpha \in (0, 1)$ such that

$$B = (1 - \alpha)I + \alpha N.$$

From $Bu = \lambda u$ it follows that

$$Nu = \frac{\lambda + \alpha - 1}{\alpha} u.$$

Since N is ne, we must have

$$\left|\frac{\lambda + \alpha - 1}{\alpha}\right| \leq 1,$$

or

$$|\lambda + \alpha - 1| \leq \alpha.$$

Since B is ne, all its eigenvalues must have $|\lambda| \leq 1$. We consider the cases of λ real and λ complex and not real separately.

If λ is real, then we need only show that we cannot have $\lambda = -1$. But if this were the case, we would have

$$|-2+\alpha| = 2-\alpha \leq \alpha,$$

or $\alpha > 1$, which is false.

If $\lambda = a + bi$, with $a^2 + b^2 = 1$ and $b \neq 0$, then we have

$$|(a+\alpha-1)+bi|^2 \leq \alpha^2,$$

or

$$(a+(\alpha-1))^2 + b^2 \leq \alpha^2.$$

Then

$$a^2 + b^2 + 2a(\alpha-1) + (\alpha-1)^2 \leq \alpha^2.$$

Suppose that $a^2 + b^2 = 1$ and $b \neq 0$. Then we have

$$1 \leq \alpha^2 + 2a(1-\alpha) - (1-\alpha)^2 = 2a(1-\alpha) - 1 + 2\alpha,$$

or

$$2 \leq 2(1-\alpha)a + 2\alpha,$$

so that

$$1 \leq (1-\alpha)a + \alpha(1),$$

which is a convex combination of the real numbers a and 1. Since $\alpha \in (0,1)$, we can say that, unless $a = 1$, 1 is strictly less than the maximum of 1 and a, implying that $a \geq 1$, which is false. Therefore, we conclude that $a^2 + b^2 < 1$.

Chapter 20

Convex Feasibility and Related Problems

20.1 A Lemma

For $i = 1, ..., I$ let C_i be a non-empty, closed convex set in R^J. Let $C = \cap_{i=1}^I C_i$ be the non-empty intersection of the C_i.

Lemma 20.1 *If $c \in C$ and $x = c + \sum_{i=1}^I p^i$, where, for each i, $c = P_{C_i}(c + p^i)$, then $c = P_C x$.*

20.2 Exercises

20.1 *Prove Lemma 20.1.*

For each i we have

$$\langle c - (c + p^i), c^i - c\rangle = \langle -p^i, c^i - c\rangle \geq 0,$$

for all c^i in C_i. Then, for any d in $C = \cap_{i=1}^I C_i$, we have

$$\langle -p^i, d - c\rangle \geq 0.$$

Therefore,

$$\langle c - x, d - c\rangle = \langle -\sum_{i=1}^I p^i, d - c\rangle = \sum_{i=1}^I \langle -p^i, d - c\rangle \geq 0,$$

from which we conclude that $c = P_C x$.