CHAPTER 2

Qualitative Solution Sketching for First-Order Differential Equations

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1. Introduction

Qualitative solution sketching, as explained in this module, can yield very useful information about the solutions $y = f(x)$ to a given differential equation $y' = g(x, y)$. In particular, it usually allows you to examine the limiting or long-range behavior for y as $x \to \infty$ without actually coming up with an explicit expression for the solution. Frequently, an explicit solution is unnecessary or technically difficult, or it might not exist at all in terms of elementary functions (polynomial, trigonometric, logarithmic, exponential). In these cases the qualitative approach may be a lifesaver—or at least a worksaver.

Recall that the differential equation $y' = g(x, y)$ has a whole *family* of solutions of the form $y = f(x)$. Different solutions in the family result from different values of the constant of integration, e.g., $y' = 1$ has a family of solutions $y = x + c$ (see Fig. 2.1). For a particular problem, the value of c is determined by some given condition $f(x_0) = y_0$; this determines which member of the family of solutions is *the* solution. However, we shall not be looking for explicit solutions, much less particular solutions, so we shall be sketching the *family* of solutions to each differential equation.

In elementary calculus you learned how derivatives could be used in graphing a function. Now we shall capitalize on those skills and show how to do it even when you do not have the explicit equation $y = f(x)$ to help you (Sections 2–4). Furthermore, we shall see that much additional information on equilibria and stability of solutions can be extracted from these sketches (Section 5).

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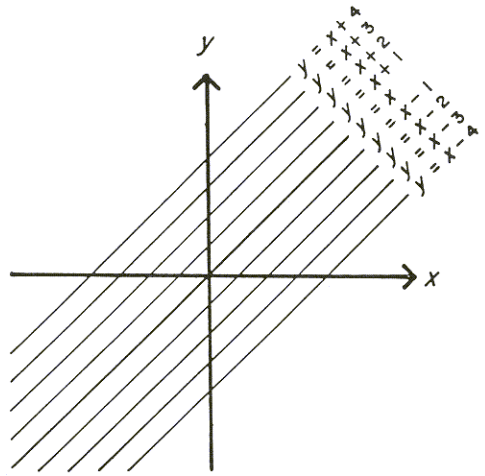


Figure 2.1

2. Direction Field

If a first-order differential equation can be put in the form $y' = g(x, y)$, then we can determine the *slope* of the solution $y = f(x)$ through any point (x, y). Graphically, we can draw a short line of the proper slope through each of many points (x, y) in the plane. This is called a *direction map* or *direction field.*

EXAMPLE 1. Consider

$$\frac{dy}{dx} = -xy. \tag{1}$$

You might start off with a few simple calculations, as tabulated here.

	x	y	$y' = \frac{dy}{dx} = -xy$
if $x = 0$ or if $y = 0$, $y' = 0$, so the direction	0	anything	0
lines along both axes are all horizontal	anything	0	0
if $x = 1$, $y' = -1$, so	1	1	−1
	1	2	−2
if $x = 1/2$, $y' = -y/2$, so	1/2	1	−1/2
	1/2	2	−1

With very little further precise calculation, you can continue to fill in graphically the direction field by noting facts as the following.

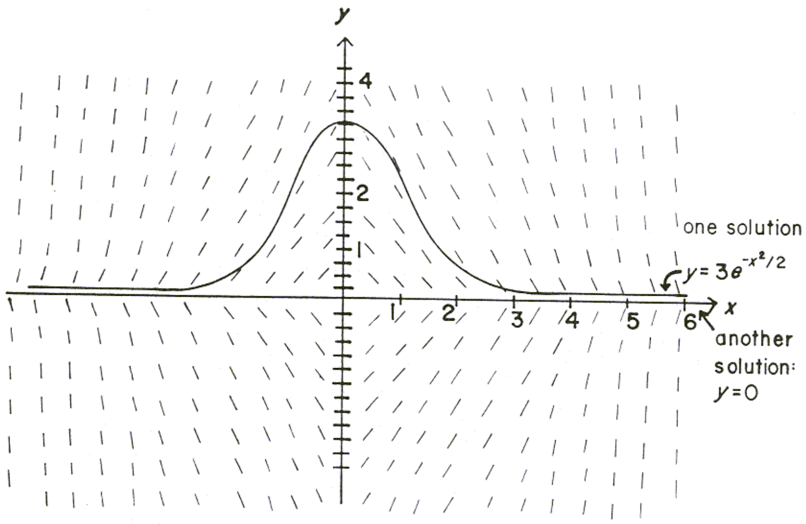


Figure 2.2

(1) Symmetry exists about the origin and about both axes, so considering the first quadrant in detail gives you all the information for the other three.
(2) For fixed x, the slopes get steeper as y increases.
(3) For fixed y, the slopes get steeper as x increases.

Thus you can arrive rather quickly at a direction field (Fig. 2.2).

A direction field such as this gives a visual indication of the family of all possible solutions to the differential equation. Any one solution must be tangent to these direction lines for each point through which it passes. Usually, the solutions can be put in the form $y = f(x)$, with each member of the family having different values of the constant which results from integration. For instance, the general solution to (1) is, in fact, $y = a e^{-x^2/2}$, and one of these solutions, with $a = 3$, has been drawn in on the direction field so that you may see how it fits all the direction lines of the map. The actual solution is easily obtained by separation of variables:

$$y' = -xy$$

$$\int \frac{dy}{y} = \int -x\,dx$$

$$\ln y = -\frac{x^2}{2} + c$$

$$y = e^{-x^2/2+c} = e^c e^{-x^2/2} = a e^{-x^2/2}.$$

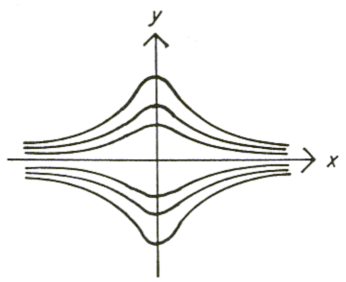


Figure 2.3

3. Relevance of Uniqueness Theorems

Whenever a uniqueness theorem applies to a first-order differential equation, there can be only *one* solution through any point (x, y), so *no two solutions can ever cross*. This is terribly helpful information when you are trying to sketch a family of solutions. For instance, in Example 1, this means the *only* way to draw in a family of solutions on that direction field is as illustrated in Figure 2.3.

You will now want to know "when does a uniqueness theorem apply?" The closest we can come to an easy answer is "usually" for first-order ordinary differential equations. To be mathematically precise is rather complicated. Any elementary text on differential equations will focus on this equation and will state some theorem such as the following.

Theorem. *For a first-order differential equation $y' = g(x, y)$ with initial condition $y(x_0) = y_0$, a sufficient (though not necessary) condition that a unique solution $y = f(x)$ exist is that g and $\partial g/\partial y$ be real, finite, single-valued, and continuous over a rectangular region of the plane containing the point (x_0, y_0).*

Seldom do elementary texts *prove* such theorems. A notable exception is Martin Braun's *Differential Equations and their Applications* (New York: Springer–Verlag, 1978, 2nd ed.).[1] In Chapter 1.10 Braun gives an excellent discussion, with proofs, of questions of uniqueness and existence. This treatment is easily accessible to anyone with a simple elementary calculus background.

In any case, you can see that the criterion for uniqueness of solutions to $y' = g(x, y)$ with a given condition is roughly that g and $\partial g/\partial y$ are "nice." We shall now look at some of the many cases where this is so.

[1] See also W. E. Boyce and R. C. Di Prima, *Elementary Differential Equations and Boundary Value Problems* (New York: Wiley, 1977, 3rd ed.)

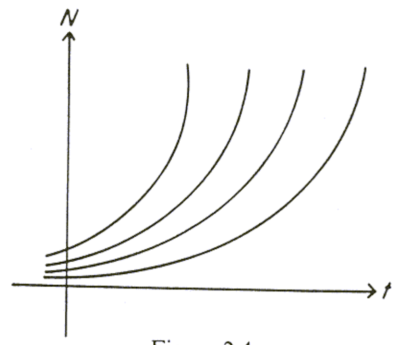


Figure 2.4

4. Sketching of Solutions

From Sections 2 and 3 we have derived two guidelines for sketching solutions of first-order differential equations $y' = g(x, y)$:

(1) solutions must be consistent with the direction field;
(2) for most of these (e.g., with g and $\partial g/\partial y$ continuous), uniqueness tells us that no two solutions will ever cross.

Keep these guidelines in mind as you consider the following example.

EXAMPLE 2. Consider the simplest population growth assumption, that the per capita rate of growth is constant. This means that if $N(t)$ = population as a function of time,

$$\frac{1}{N}\frac{dN}{dt} = r = \text{const.} \tag{2}$$

To get a quick qualitative picture of possible solutions, rewrite as

$$\frac{dN}{dt} = rN$$

and consider the graphical behavior of $N(t)$.

For this physical problem, we need only $N > 0$, $t > 0$. You can see that slope $= dN/dt = rN = 0$ when $N = 0$, and that slope increases as N increases. Presto, you have shown that the solutions must look *approximately* like Figure 2.4.

This rough sketch is quite satisfactory for showing clearly that for *all* solutions, N rises ever faster, with no end in sight. You can easily confirm the sketch by solving (2) to get $N = N_0 e^{rt}$.

We note two additional facts which will give a more accurate graphical picture and clarify the role of the constants r and N_0: the effect of r is to increase the slope as r increases, at a given N (Fig. 2.5). In this chapter we want to consider the constants in the differential equation as fixed and to sketch only the family of solutions arising from different constants of in-

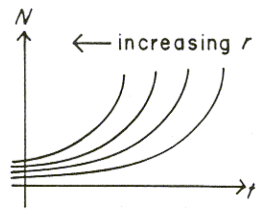


Figure 2.5

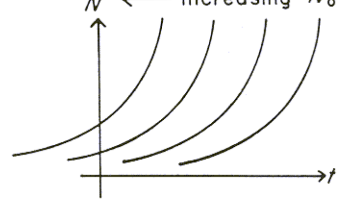


Figure 2.6

tegration. For a *given fixed* r, look at a given N. Since no t is explicit in dN/dt, the *same* slopes occur at that N for all t. Hence all solution curves are horizontal translates of one another (Fig. 2.6).

EXAMPLE 3. Consider qualitatively what happens when the basic assumption of (2) in the above example is modified to show the decrease in per capita rate of growth due to *crowding*. The simple Verhulst–Pearl expression of this modification is

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{k}\right). \tag{3}$$

Here the per capita rate $(1/N)/(dN/dt)$ decreases linearly, diminished by a crowding term directly proportional to N, the size of the population. The positive constant r still represents the per capita rate of growth when no crowding occurs (when $N = 0$).

To look at the solutions graphically, we rewrite

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) = \text{slope}.$$

We can see that slope is zero (horizontal) for $N = 0$ and for $N = k$. We can also show that the slope is positive for $0 < N < k$ and negative for $N > k$, but that does not seem to be enough information to tell us how to graph solutions that do not cross (for uniqueness). See Figure 2.7.

Our other graphing trick from calculus is the determination of *concavity*, which depends on the *second* derivative. This we can get:

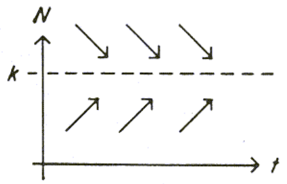


Figure 2.7

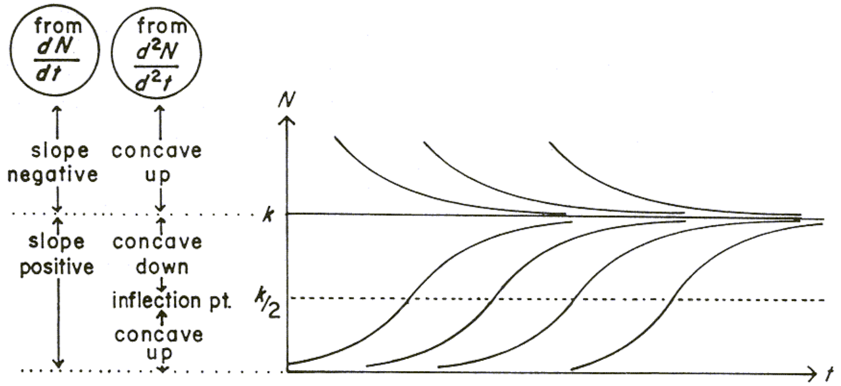


Figure 2.8

$$\frac{d^2N}{dt^2} = \frac{d}{dt}\frac{dN}{dt} = r\frac{dN}{dt}\left(1 - \frac{N}{k}\right) + rN\left(-\frac{1}{k}\frac{dN}{dt}\right)$$

$$= r\frac{dN}{dt}\left(1 - \frac{N}{k} - \frac{N}{k}\right) = r\frac{dN}{dt}\left(1 - \frac{2N}{k}\right).$$

We want to know where the second derivative is 0, positive, and negative. Considering dN/dt and $(1 - (2N/k))$ as the two factors (multiplied by positive r), we can show that $d^2N/dt^2 = 0$ when $N = 0$, k, or $k/2$, that it is positive (N concave upward) for $N < k/2$ and for $N > k$, and that it is negative (N concave downward) for $k/2 < N < k$.
All this information fits together as illustrated in Figure 2.8.

No t is explicit in dN/dt, so again (for fixed r and k), the *same* slopes occur at a given N for all t. Hence all solution curves are horizontal translates of one another.

Because (3) cannot be solved by simple integration after separation of variables, we are already into the area where our sketching technique becomes valuable. (Equation (3) *can* be solved by partial fraction integration, or by making the substitution $M = 1/N$ which gives a linear first-order differential equation for M. Exercise 22 also illustrates a surprisingly good

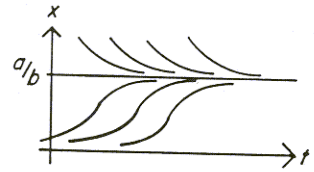


Figure 2.9

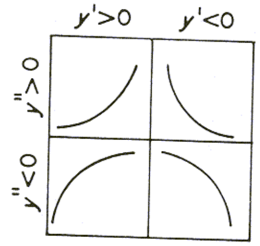


Figure 2.10

fit of this model to actual population data of the United States from 1790 to 1910.)

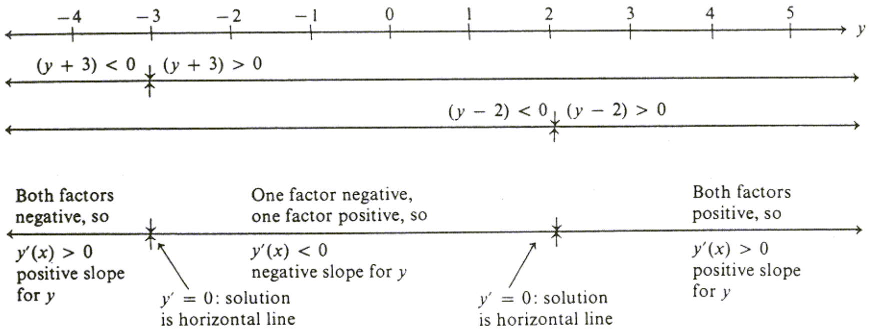
In general, equations of the form of (3) arise frequently and are called *logistic*. This means that any equation of the form $dx/dt = x(a - bx)$ is called logistic and is known to have solutions of the form illustrated in Figure 2.9.

These examples lead us to the formulation of a general approach. We want to combine information from the first and second derivatives to graph solutions which exhibit approximately the proper slope and concavity. (This process has been likened to the construction of a composite sketch of a crime suspect from a Dragnet Idento-Kit. You can consider the drawing below as a solution-sketching idento-kit.) The four basic types of curves are shown in Figure 2.10. The problem is simply to determine which pieces go where in the coordinate plane. The following example illustrates a good procedure.

Example 4. Sketch a number of solutions to the following differential equation. Your sketches should exhibit approximately the right slope and concavity,

$$y' = (y + 3)(y - 2). \tag{4}$$

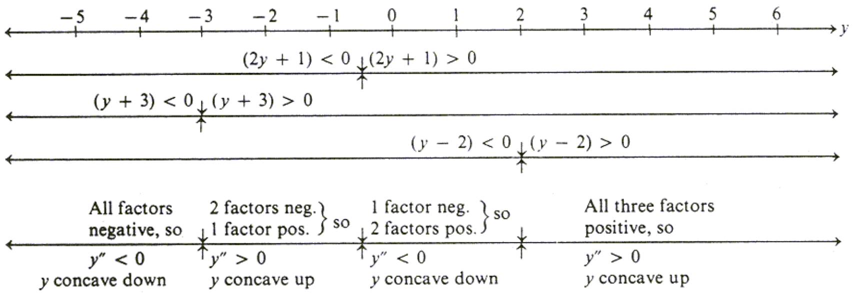
1) *Look at y' (slope)*: $y' = (y + 3)(y - 2)$ is a product of two factors, so we need to determine the positivity and negativity of the factors. You can keep this straight on a number line (for y), as shown.



2) *Look at y″ (concavity)*: Using implicit differentiation and the product rule on (4), we get

$$y''(x) = y'(y - 2) + (y + 3)y'$$
$$= (2y + 1)y'$$
$$= (2y + 1)(y + 3)(y - 2).$$

Use the number line again to deal with this product of three factors.



3) *Put it all together*: Sketch in the horizontal solutions $y = -3$ and $y = 2$. Use the information from 1) and 2) to block off the coordinate plane wherever changes occur in y' or y''—in this case, at $y = -3$, $y = -1/2$, $y = 2$. Then put in the right curve pieces. Figure 2.11 results.

Here are some notes on the above solution sketches for Example 4.

a) Where the ⟍ piece meets the ⟍_ piece, an inflection point occurs.

b) The solution curves must approach the equilibria asymptotically because you cannot have two solutions passing through the same point (i.e., no two solution curves can intersect).

c) Again, no x is explicit in y', so horizontal translates of solution curves are also solution curves.

d) Using the differential equation (4), we *could* calculate the slope exactly

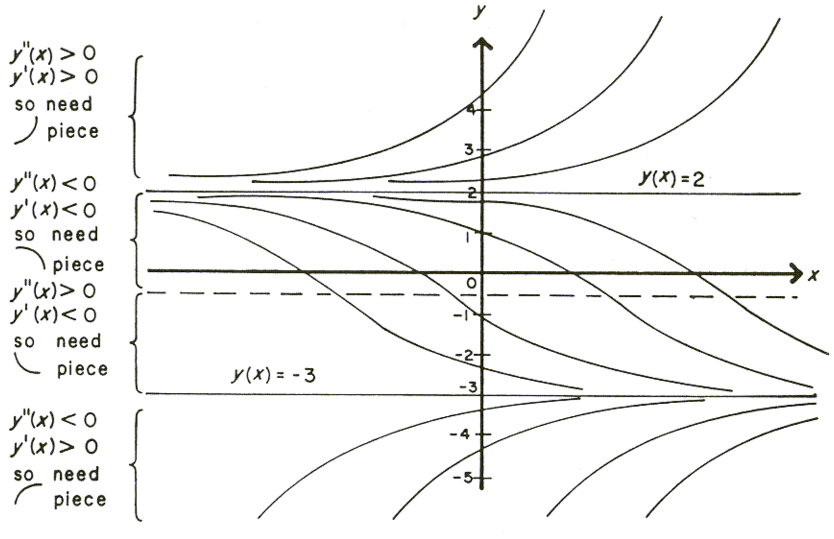


Figure 2.11

at any point if we so desire, but we are able to get this beautiful picture even without a point-by-point plot of the direction field.

You may streamline the details of this last example in your own work, but headings 1), 2), and 3) list the key steps to obtaining good solution sketches. In the next section we shall examine these sketches to see what they tell us.

5. Discussion of Equilibria and Stability

An *equilibrium* solution to a differential equation is one that *does not change* or *is constant*. That means an equilibrium is a solution for which $y' = g(x, y) \equiv 0$ or $y = f(x) \equiv c$, for all real numbers x. In Example 4, the solutions $y = -3$ and $y = 2$ are equilibrium solutions.

Equilibria may be classified as stable or unstable. The next illustration may help in distinguishing the two. Consider two cones, one sitting on its base, and the other balancing on its point (Fig. 2.12). Both cones are in equilibrium: their positions are not changing. If you jiggle the left cone a bit with your finger, it will quickly return to its original position. This is a *stable* equilibrium. If you jiggle the right-hand cone, however, it will promptly fall down, demonstrating that the equilibrium was *unstable*.

Equilibrium solutions to differential equations are classified as stable or unstable depending on whether, graphically, nearby solutions converge to the equilibrium or diverge from the equilibrium, respectively, *as* $x \to \infty$.

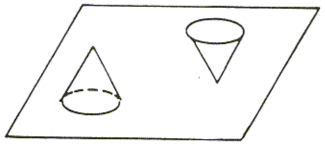


Figure 2.12

The graph of Example 4 demonstrates a stable equilibrium at $y = -3$ and an unstable equilibrium at $y = 2$. What does this mean physically? A possible interpretation of this unstable equilibrium is given by the following. Suppose $y(x)$ represents millions of bacteria on a plate at time x. Suppose you have a plate with 2,000,000 bacteria at time x_0. Then you are on the $y(x) = 2$ equilibrium solution. If someone sneezes on the plate at time x_0, suddenly giving you 2,000,126 bacteria, you now go on to one of the $y' > 0$, $y'' > 0$ solutions, and the number of bacteria increases drastically. Similarly, if you had removed 617 of the bacteria at time x_0, your population of 1,999,383 would now be represented by one of the $y' < 0$, $y'' < 0$ curves, and the number of bacteria will decrease as time passes. So, when we jiggle or *perturb* this equilibrium solution $y = 2$, the conditions change drastically. The new situation never returns to $y = 2$.

On the other hand, if a system is perturbed near a stable equilibrium, such as $N = k$ of Example 3, it tends to come right back to the equilibrium. A population of $N = k$ would tend to remain at $N = k$, and one with N close to k will move toward the level $N = k$. This question of stability is crucial in many applied problems (see Exercises 5–8, 17, 18).

Now it is time for *you* to try some problems. Again, the point of this chapter is to teach *you* how to do it, not to show that *some* people can. Try some of the Exercises 1–19. Talk with someone else if you get stuck. *When* you feel you have conquered some of those, try the following.

6. A More Difficult Example

EXAMPLE 5. Sketch solutions for

$$y' = 2x - y. \tag{5}$$

Try this problem before reading further. *Then* the following explanations are more apt to clarify and less apt to confuse or complicate your thinking.

1) *Look at* y':

$$y' = 0, \quad \text{for} \quad y = 2x$$
$$y' > 0, \quad \text{for} \quad y < 2x$$
$$y' < 0, \quad \text{for} \quad y > 2x$$

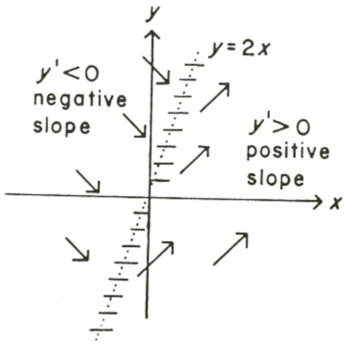


Figure 2.13

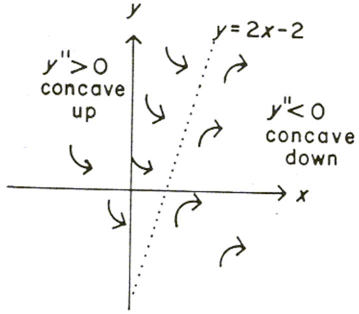


Figure 2.14

which gives us, so far, Figure 2.13. (Note: Where $y' = 0$, y is not a constant. Therefore, there is no equilibrium, no horizontal solution.)

2) *Look at* y'':

$$y'' = 2 - y' = 2 - 2x + y$$
$$y'' = 0, \text{ for } y = 2x - 2$$
$$y'' > 0, \text{ for } y > 2x - 2$$
$$y'' < 0, \text{ for } y < 2x - 2$$

giving us, in addition to the information in 1), Figure 2.14.

3) *Put it together*:

y	y'	y''	Conclusion
Greater than (to the left of) $y = 2x$	−	+	
Between $y = 2x$ and $y = 2x - 2$	+	+	
Less than (to the right of) $y = 2x - 2$	+	−	

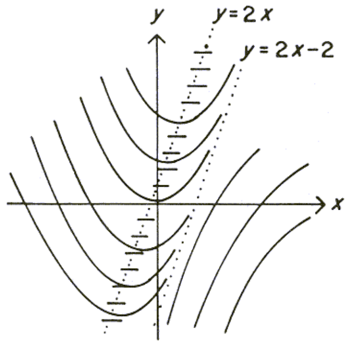


Figure 2.15

The tabular results are graphed in Figure 2.15. Remember, it all has to fit together. Further questions arise before you can be sure you have finished.

a) The solutions on the left of the last diagram will fit without crossing and destroying uniqueness *if* $y = 2x - 2$ is an *asymptote*. Is it? Recall that an asymptote is a line which is approached, but never reached, as $x \to \infty$. If $y = 2x - 2$, then $y' = 2$, and the original equation (5) is also satisfied. So, $y = 2x - 2$ is a solution, with constant slope 2, and the other solutions can never cross it.

b) What exactly happens to all those positive-slope, concave-downward solutions on the right of $y = 2x - 2$? How quickly do they bend over? Do they approach a limit as $x \to \infty$?

At this point we would be smart to go back to the direction field. We know where $y' = 0$ (along $y = 2x$) and where $y' = 2$ (along $y = 2x - 2$). From there, it is easy to consider

a fixed y since $y' = 2x - y$, as $x\uparrow$, $y'\uparrow$ (slopes get steeper as you move to the right along a fixed y);

a fixed x since $y' = 2x - y$, as $y\uparrow$, $y'\downarrow$ (slopes diminish as you move higher along a vertical line).

Alternatively, you might note that along any line $y = 2x + k$, the slope $y' = k$.

So, putting all this together also, we can improve the sketch in the final result, note that $y = 2x - 2$ is an asymptote for the solutions on the right as well. The sketch of solutions to (5)

$$y' = 2x - y$$

looks like Figure 2.16 (negative C's give the solutions below the asymptote).

The line $y = 2x - 2$, which is an asymptote, can be considered an equilibrium solution in a generalized sense. Solutions near this line tend toward it as $x \to \infty$. Actual solution of this differential equation (5) yields

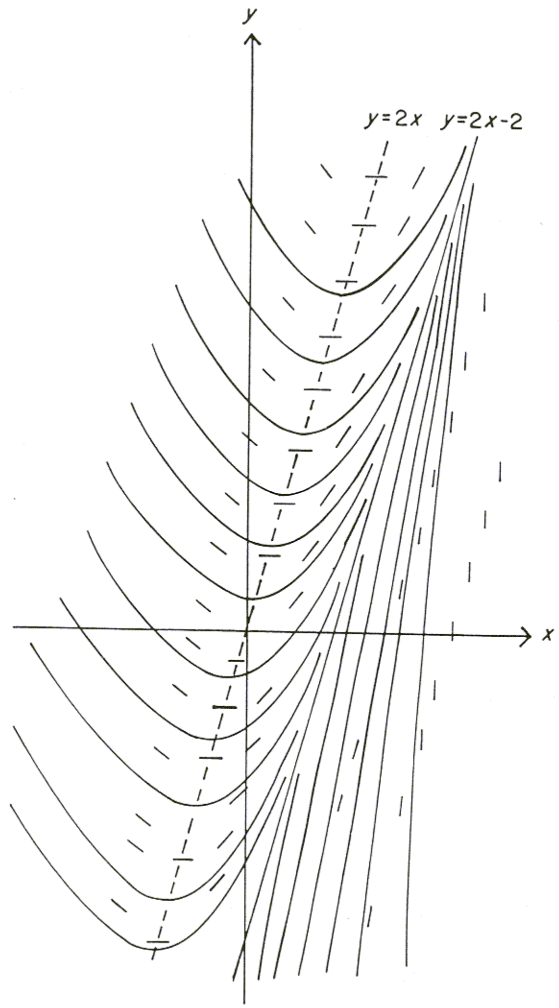


Figure 2.16. Negative c's give the solutions below the asymptote.

$$y = Ce^{-x} + 2x - 2.$$

Sure enough, as $x \to \infty$, $Ce^{-x} \to 0$. You can note that when x is quite negative, the solution is mostly Ce^{-x} (i.e., y is dominantly exponential); when x is quite positive, the solution is nearly $2x - 2$ (a straight line). The "interesting" part is where these two run into each other.

Example 5 has shown that such an innocent-looking differential equation as $y' = 2x - y$ can have a complex picture of the family of solutions. Nevertheless, that picture can be obtained simply by following through on the y', y'', put-together technique until all the pieces do fit together and everything is nailed down.

Exercises

(The most challenging exercises are denoted by a dagger.)

For Exercises 1–4, sketch solution to the given differential equations.

1. $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

2. $\dfrac{dy}{dx} = y$.

3. $x + y' - 1 = -yy'$.

4. $y' = \dfrac{2y}{x}$.

5. Nutrients flow into a cell at a constant rate of R molecules per unit time and leave it at a rate proportional to the concentration, with constant of proportionality K. Let N be the concentration at time t. Then the mathematical description of the rate of change of nutrients in the above process is

$$\frac{dN}{dt} = R - KN;$$

that is, the rate of change of N is equal to the rate at which nutrients are entering the cell minus the rate at which they are leaving. Will the concentration of nutrients reach an equilibrium? If so, what is it and is it stable? Explain, using a graph of the solutions to this equation.

6. Suppose that the average new professor at a university begins checking books out of the library at the rate of one per day. Suppose further that the library recalls, in an average *week*, 1/10 of the books checked out. How many books does the average professor than have checked out at any one time after he has been around several years?

7. Suppose that an island is colonized by immigration from the mainland. Suppose further that there are S species on the mainland and, at time t, $N(t)$ on the island. The rate that new species immigrate to the island and colonize is proportional to the number of species on the mainland which are not already established on the island $(S - N(t))$, with constant of proportionality I. Moreover, on the island species become extinct at a rate proportional to the number of species on the island, with constant of proportionality E. Show that the number of species on the island will reach an equilibrium number approximately $= [I/(I + E)]S$. Sketch the curve of N as a function of t.

8. Suppose a hard rock fan club starts out at time $t = 0$ with N_0 fanatical members. The club would no doubt grow at a rate proportional to the membership, except there are at most M people who are at all interested in such music. Hence as the membership approaches M the rate decreases because new recruits become harder to find. So in actuality, the rate of increase is proportional to the product of the number of members and the number of remaining interested people. Give the differential equation involving membership, $N(t)$. How many memberships can the organizers expect to sell per year, assuming the constants remain steady for a few years?

Sketch a number of solutions to the following equations. Your sketches should exhibit the correct slope and concavity and indicate points of inflection.

9. $y' = y^2 - 1$.

10. $y' = y(y - 2)(y - 4)$.

11. $y' = (e^{-x} - 1)y$.

12. $y' = (y - 1)(3 + y)$. Include the solution where $y(0) = 1.5$.

13. $dy/dx = x(2 - y)/(x + 1)$, $x > 0$. Find all equilibrium points, and classify them as stable or unstable. Sketch the solutions which cross the y axis at 0, 1, 2, and 3. Make sure your curves have the correct slope at $x = 0$. However, y'' is very complicated to find, so you might try to do without looking at it.

14. Sketch a solution to the equation $y' + y^4 = 16$ when $y(0) = 0$. Find the equilibrium values, if any, and tell whether they are stable or unstable.

15. Sketch a solution to the equation $y' + y^3 = 8$ when $y(0) = 0$. Find equilibrium values, if any, and tell whether they are stable or unstable.

16. Without solving the differential equation $y' = x + y$, sketch a solution between $x = 0$ and $x = 1$ satisfying $y(0) = 0$. Make sure your curve has the correct slope at the origin and the correct concavity.

17. Water flows into a conical tank at a rate of k_1 units of volume per unit time. Water evaporates from the tank at a rate proportional to $V^{2/3}$, where V is the volume of water in the tank. Let the constant of proportionality be k_2. Find the differential equation satisfied by V. Without solving it, sketch some solutions. Is there an equilibrium? Is it stable?

18. A population of bugs on a plate tend to live in a circular colony. If N is the number of bugs and r_1 is the per capita growth rate, then $dN/dt = r_1 N$ is the Malthusian growth rule. However, those bugs on the perimeter suffer from cold, and they die at a rate proportional to their number, which means that they die at a rate proportional to $N^{1/2}$. Let this constant of proportionality be r_2. Find the differential equation satisfied by N. Without solving it, sketch some solutions. Is there an equilibrium? If so, is it stable?

19. Let $dy/dx = y((1/x) - 1)$, $x \geq 1$. Find all equilibrium points and classify them as stable or unstable. Sketch solutions with $y(1) = 1$ and $y(1) = -1$. Indicate points of inflection. Your curves should have the correct slope at $x = 1$ and the correct concavity. For a general solution $y(x)$, what happens as $x \to \infty$?

†20. Consider the differential equation $y' = e^x - y$. Without solving this equation, sketch solutions passing through the following points: $(0, 1)$, $(1, e)$, $(-1, 1/e)$, $(0, 0)$, $(1, 0)$, $(-1, 0)$. In addition to these curves and coordinate axes, sketch $y = e^x$. Your curves should exhibit the correct slope at the given points and the correct concavity. You should write down your reasoning. Do you see any sort of stable solution?

†21. Sketch the curve $y(x) = \int_0^x e^{-t^2}\, dt$, paying careful attention to slope, concavity, and initial value $y(0)$. *Hint:* Use the Fundamental Theorem of Calculus to notice that $y' = e^{-x^2}$.