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A countable set of directions is sufficient for Steiner symmetrization

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ABSTRACT

A countable dense set of directions is sufficient for Steiner symmetrization, but the order of directions matters.

0. Introduction and background

Denote n-dimensional Euclidean space by $\mathbb{R}^n$, and let K be a compact convex subset of $\mathbb{R}^n$. Given a unit vector u, view K as a family of line segments parallel to u. Slide these segments along u so that each is symmetrically balanced around the hyperplane $u^\perp$. By Cavalieri's principle, the volume of K is unchanged by this rearrangement. The new set, called the *Steiner symmetrization* of K in the direction of u, will be denoted by $\mathrm{s}_u K$. It is not difficult to show that $\mathrm{s}_u K$ is also convex, and that $\mathrm{s}_u K \subseteq \mathrm{s}_u L$ whenever $K \subseteq L$. A little more work verifies the following intuitive assertion: if you iterate Steiner symmetrizations of K through a suitable sequence of unit directions, the successive Steiner symmetrals of K will approach a Euclidean ball in the Hausdorff topology on compact (convex)

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subsets of $\mathbb{R}^n$. A detailed proof of this assertion can be found in any of [8, p. 98], [14, p. 172], or [29, p. 313], as well as in Section 2 below.

Questions remain surrounding the following issue: Given a convex body K, under what more specific conditions on the sequence of directions u_i does the sequence of Steiner symmetrals

$$\mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K \tag{1}$$

converge to a ball? For example, Mani [21] has shown that, given a sequence of unit directions u_i chosen uniformly at random, the sequence (1) converges to a ball almost surely; that is, with unit probability. An explicit algorithm for rounding out a convex body with a periodic sequence of Steiner symmetrizations is described by Eggleston [8, p. 98].

For over 150 years Steiner symmetrization has been a fundamental tool for attacking problems regarding isoperimetry and related geometric inequalities [11,12,25,26]. Steiner symmetrization appears in the titles of dozens of papers (see e.g. [2–7,9,10,17–19,21,22,24]) and plays a key role in recent work such as [15,20,27,28]. In spite of the ubiquity of Steiner symmetrization throughout geometric analysis, many elementary questions about this construction remain unanswered. The authors of a recent paper [20] required a sequence of Steiner symmetrizations that rounded out a given convex body, using only directions drawn from a restricted dense set of directions in the unit sphere. Is *any* dense set of directions sufficient?

While this more subtle fact may be derived from known results of a more highly technical nature (such as recent work of Van Schaftingen [27,28]), it is not explicitly stated in the literature. We give a very simple proof that is a variation of known proofs of the standard Steiner symmetrization convergence theorem (such as that given in [29]). Along the way, we also address the related open question: If a given countable dense set of directions can be used to round out a body K, will this set always work, regardless of its *ordering* when arranged in a sequence?

More precisely, suppose we are given a set of directions Ω that is dense in the unit sphere. Is it always possible to choose directions u_i from this *restricted* set Ω so that the sequence (1) converges to a ball? The short answer is *Yes*, provided the directions are chosen (and ordered) carefully. On the other hand, it turns out that an arbitrary countable dense sequence of directions may *fail* to accomplish this; that is, the ordering of the directions could make a difference. In Section 1 we show by explicit example that, for certain orderings of the directions u_i, the limit of the sequence (1) may *fail to exist*. Then, in Section 2, we give an elementary proof that, given a convex body K and a suitably ordered choice of directions u_i from a dense set Ω, the sequence (1) converges to a ball.

1. Not every choice works

In this section we construct a dense sequence of directions in the unit circle whose corresponding sequence of Steiner symmetrizations *fails to converge* on a substantial family of convex bodies. While the example is given in dimension 2, it can be easily generalized to arbitrary finite dimension. The example that follows demonstrates the need for care when iterating Steiner symmetrization as an infinite process.

Let $\{p_1, p_2, \ldots\}$ denote the sequence of positive prime integers. Recall that the sum

$$\sum_{i=1}^{\infty} \frac{1}{p_i} \tag{2}$$

diverges [1, p. 18]. For $m \geqslant 1$, let u_m denote the unit vector in $\mathbb{R}^2$ having counter-clockwise angle

$$\theta_m = \sum_{i=1}^{m} \frac{\sqrt{2}}{p_i}$$

with the horizontal axis, measured in radians. Since $\theta_m \to \infty$, while each successive incremental angle $\frac{\sqrt{2}}{p_m} \to 0$, the unit vectors u_m form a countable dense subset of the unit circle.

Meanwhile, observe that

$$\prod_{i=1}^{\infty} \cos\left(\frac{\sqrt{2}}{p_i}\right) \geqslant \prod_{i=1}^{\infty}\left(1 - \frac{1}{p_i^2}\right).$$

Applying the Euler product formula [16, p. 246], we obtain

$$\left(\prod_{i=1}^{\infty} \cos\left(\frac{\sqrt{2}}{p_i}\right)\right)^{-1} \leqslant \prod_{i=1}^{\infty}\left(\frac{1}{1 - \frac{1}{p_i^2}}\right) = \prod_{i=1}^{\infty}\left(1 + \frac{1}{p_i^2} + \frac{1}{p_i^4} + \cdots\right) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

so that

$$\prod_{i=1}^{\infty} \cos\left(\frac{\sqrt{2}}{p_i}\right) \geqslant \frac{6}{\pi^2}. \tag{3}$$

Let ℓ be a vertical line segment, centered at the origin, of length 1. Apply the sequence of Steiner symmetrizations s_{u_m} to ℓ. Each symmetrization has the effect of projecting the previous line segment onto the line perpendicular to u_m, thereby multiplying the previous length by the next incremental cosine, $\cos(\frac{\sqrt{2}}{p_m})$. Since the limiting value of the product (3) is strictly positive (greater than 1/2, in fact), while the angles θ_m cycle around the circle forever, the iterated Steiner symmetrals of ℓ also spin in circles forever, while approaching a limiting positive length.

In particular, the sequence of line segments

$$\ell_m = \mathrm{s}_{u_m} \cdots \mathrm{s}_{u_1} \ell$$

has no limit.

Now let K be a cigar-shaped convex body of area ε containing ℓ as an axis of symmetry. By the monotonicity of Steiner symmetrization, each element in the sequence of Steiner symmetrals

$$K_m = \mathrm{s}_{u_m} \cdots \mathrm{s}_{u_1} K$$

must contain the corresponding symmetral ℓ_m, so that the diameter of each K_m exceeds $\frac{6}{\pi^2}$. Since each K_m has the same area ε as the original body K, which could be made arbitrarily small beforehand, it follows that the sequence K_m cannot approximate a ball. Indeed, for $\varepsilon < \frac{9}{\pi^3}$ the sequence K_m has no limit, since the diameter line revolves forever, but does not shrink enough to accommodate the tiny given area ε.

We have shown that a countable dense sequence of directions does not necessarily lead to a well-defined limiting Steiner symmetral.

In this specific example we used the divergent series (2) as a starting point for computational convenience. Gronchi [13] has shown that a more general family of examples can be constructed starting with any decreasing sequence of incremental angles θ_i provided that $\sum_{i=1}^{\infty} \theta_i^2$ converges and $\sum_{i=1}^{\infty} \theta_i$ diverges. Iterated Steiner symmetrization in the resulting sequence of directions, applied to a sufficiently eccentric ellipse, results in a sequence of ellipses whose principal axes rotate forever without converging to a circle.

2. There is always an order of directions that works

In view of the previous example, it is necessary to show that, given a countable dense set of directions in $\mathbb{R}^n$, it is indeed possible to construct a sequence of directions from this set so that successive Steiner symmetrals of a given convex body K converge to a Euclidean ball.

Let Ω be a dense subset of the Euclidean unit sphere $\mathbb{S}^{n-1}$. If a convex body K in $\mathbb{R}^n$ has volume zero, then K lies in a proper subspace of $\mathbb{R}^n$. Steiner symmetrization of K in any direction outside the affine hull of K has the same effect as orthogonal projection. One can choose directions close to, but outside, the affine hull so as to shrink K by any positive factor, along any direction inside the affine hull. A suitable iteration will shrink the diameter of successive symmetrizations (projections) to zero, so that symmetrals converge to a point (a Euclidean ball of radius zero).

For a convex body K in $\mathbb{R}^n$ of positive volume, let $r_K = \max_{x \in K} |x|$, which is the minimal radius of balls centered at the origin that contain K. Let r_1 be the infimum of all r_C, where C is obtained from finitely many successive Steiner symmetrizations of K in directions that belong to Ω. Then there is a sequence of such convex bodies C_i so that $r_{C_i} \to r_1$. Obviously, the sequence $\{C_i\}$ is bounded, because each $C_i \subseteq r_K B$, where B is the unit ball. By the Blaschke selection theorem [23,29], there is a subsequence C_{i_k} that converges to a convex body K_1, where $r_{K_1} = r_1$. Denote $r_1 B$ by B_1, so that $K_1 \subseteq B_1$.

We claim that $K_1 = B_1$. Assume it is not true. There is a small cap U on ∂B_1 so that $U \cap K_1 = \emptyset$. For any line ξ such that $\xi \cap U \neq \emptyset$, either $\xi \cap K_1 = \emptyset$ or the line ξ intersects a longer chord in B_1 than in K_1; that is, $|\xi \cap B_1| > |\xi \cap K_1|$. After taking a Steiner symmetrization $\mathrm{s}_u K_1$ for some $u \in \Omega$, the symmetral $\mathrm{s}_u K_1$ fails to intersect both U and a new cap U' given by the reflection of U with respect to the hyperplane $u^\perp$. Since Ω is dense in $\mathbb{S}^{n-1}$, one can continue to take symmetrizations with respect to an appropriate finite family of hyperplanes with normals $v_1, \dots, v_s \in \Omega$ that generate finitely many caps covering the whole sphere ∂B_1 and generate a convex body K_2 so that $|\xi \cap B_1| > |\xi \cap K_2|$ for any line such that $\xi \cap \partial B_1 \neq \emptyset$. Thus, $r_{K_2} < r_1$.

Denote $\tilde{C}_{i_k} = s_{\mathrm{v}_s} \cdots s_{\mathrm{v}_1} C_{i_k}$. Since $C_{i_k} \to K_1$, while Steiner symmetrization is continuous on convex bodies with nonempty interior [29, p. 312], we have

$$\tilde{C}_{i_k} = s_{\mathrm{v}_s} \cdots s_{\mathrm{v}_1} C_{i_k} \to s_{\mathrm{v}_s} \cdots s_{\mathrm{v}_1} K_1 = K_2.$$

Since $r_{\tilde{C}_{i_k}} \to r_{K_2}$, it follows from the definition of r_1 that $r_{K_2} \geqslant r_1$, a contradiction.

We have shown that for any convex body K of positive volume there are $u_1, \dots, u_{i_1} \in \Omega$ so that the Hausdorff distance d between $\mathrm{s}_{u_{i_1}} \cdots \mathrm{s}_{u_1} K$ and the centered ball B_1 with the same volume of K can be arbitrarily small.

Denote by $d(K_1, K_2)$ the Hausdorff distance between compact convex sets $K_1, K_2 \subseteq \mathbb{R}^n$. For a sequence of positive numbers $\varepsilon_k \to 0$, there are $u_1, \dots, u_{i_1} \in \Omega$ so that $d(D_1, B_1) < \varepsilon_1$, where $D_1 = \mathrm{s}_{u_{i_1}} \cdots \mathrm{s}_{u_1} K$. Similarly, there are $u_{i_1+1}, \dots, u_{i_2} \in \Omega$ so that $d(D_2, B_1) < \varepsilon_2$, where $D_2 = \mathrm{s}_{u_{i_2}} \cdots \mathrm{s}_{u_{i_1+1}} D_1$. In general, there are $u_{i_{k-1}+1}, \dots, u_{i_k} \in \Omega$ so that $d(D_k, B_1) < \varepsilon_k$, where $D_k = \mathrm{s}_{u_{i_k}} \cdots \mathrm{s}_{u_{i_{k-1}+1}} D_{k-1}$. Since $d(\mathrm{s}_u K, B_1) \leqslant d(K, B_1)$ for any K when $d(K, B_1) < r_{B_1}$, there is a sequence $K_i = \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K \to B_1$, where $u_i \in \Omega$.

3. Related open questions

In Section 1 we described a convex body K and a sequence of directions u_i for which the sequence of Steiner symmetrals

$$K_i = \mathrm{s}_{u_i} \cdots \mathrm{s}_{u_1} K$$

failed to converge in the Hausdorff topology. However, some of the examples described in Section 1 clearly converge in *shape*: there is a corresponding sequence of isometries ψ_i such that the sequence $\{\psi_i K_i\}$ converges. Is this always the case? If so, and supposing also that the sequence $\{u_i\}$ is dense in the unit sphere $\mathbb{S}^{n-1}$, is the limit of the convergent sequence $\{\psi_i K_i\}$ always an ellipsoid? Moreover, what happens if K is permitted to be an arbitrary (possibly non-convex) compact set?

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