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VOLUME BOUNDS FOR SHADOW COVERING

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ABSTRACT. For $n \geq 2$ a construction is given for a large family of compact convex sets K and L in $\mathbb{R}^n$ such that the orthogonal projection L_u onto the subspace $u^\perp$ contains a translate of the corresponding projection K_u for every direction u, while the volumes of K and L satisfy $V_n(K) > V_n(L)$.

It is subsequently shown that if the orthogonal projection L_u onto the subspace $u^\perp$ contains a translate of K_u for every direction u, then the set $\frac{n}{n-1}L$ contains a translate of K. It follows that

$$V_n(K) \leq \left(\frac{n}{n-1}\right)^n V_n(L).$$

In particular, we derive a *universal constant bound*

$$V_n(K) \leq 2.942\, V_n(L),$$

independent of the dimension n of the ambient space. Related results are obtained for projections onto subspaces of some fixed intermediate co-dimension. Open questions and conjectures are also posed.

1. INTRODUCTION

Suppose that K and L are compact convex subsets of n-dimensional Euclidean space. For a given fixed dimension $1 \leq k < n$, suppose that every k-dimensional orthogonal projection (shadow) of K can be translated inside the corresponding projection of L. How are the volumes of K and L related? Also, under what additional conditions does it follow that L contains a translate of K?

Several aspects of this problem have been recently addressed in [19, 20, 21]. In [20] it was shown that, despite the assumption on covering by all k-dimensional projections, it may still be the case that K has *greater* volume than L.

It is also shown in [20] that if the orthogonal projection L_u of L onto the $(n-1)$-dimensional subspace $u^\perp$ contains a translate of the corresponding projection K_u for every unit direction $u \in \mathbb{R}^n$, then the volumes must satisfy $V_n(K) \leq nV_n(L)$, and that $V_n(K) \leq V_n(L)$ if L can be approximated by Blaschke combinations of convex cylinders in $\mathbb{R}^n$. Earlier results of Ball [2] imply that the covering condition on projections (as well as the much weaker condition that projections of K have smaller area) imply that the volume ratio $\frac{V_n(K)}{V_n(L)}$ is bounded by a function that grows with order $\sqrt{n}$ as the dimension n becomes large. However, all specific examples so far computed have suggested that the volume ratio is much smaller.

In this article we prove that the volume of K, while possibly exceeding that of L, must still always satisfy

$$V_n(K) \leq \left(\frac{n}{n-1}\right)^n V_n(L).$$

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In particular, there is a *universal constant bound*

$$V_n(K) \leq 2.942\, V_n(L), \tag{1.1}$$

independent of the dimension n of the ambient space (see Section 5).

This constant bound will be seen as the direct consequence of the Main Theorem of this article:

Main Theorem. *Let K and L be compact convex sets in $\mathbb{R}^n$. Suppose that, for every unit vector u, the orthogonal projection L_u of L onto the subspace $u^\perp$ contains a translate of the corresponding projection K_u. Then there exists $x \in \mathbb{R}^n$ such that*

$$K + x \subseteq \left(\tfrac{n}{n-1}\right) L.$$

We also provide a substantial source of examples of compact convex sets K and L such that the volume ratio is strictly greater than 1, adding to the special case described in [20].

The background material for these results is described in Section 2. In Section 3 we describe a large family of convex bodies K and L such that each projection of L contains a translate of the corresponding projection of K, while K has greater volume.

In Section 4 we show that if the body L having larger projections is a *simplex*, then there is a translate of K that lies inside a cap body of L having volume $\frac{n}{n-1} V_n(L)$. Section 5 combines the simplicial case with a containment theorem of Lutwak [26] (see also [22, p. 54]) to prove the Main Theorem. The universal constant bound (1.1) for volume ratios is then derived as a corollary.

Section 6 extends the results of the previous sections to the case in which $(n-d)$-dimensional projections of L contain translates of $(n-d)$-dimensional projections of K for some intermediate co-dimension $1 \leq d \leq n-1$. In Section 7 we pose some open questions and conjectures.

This investigation is motivated in part by the projection theorems of Groemer [15], Hadwiger [18], and Rogers [29]. In particular, if two compact convex sets have translation congruent (or, more generally, homothetic) projections in every linear subspace of some chosen dimension $k \geq 2$, then the original sets K and L must be translation congruent (or homothetic). Rogers also proved analogous results for sections of sets with hyperplanes through a base point [29]. These results then set the stage for more general (and often much more difficult) questions, in which the rigid conditions of translation congruence or homothety are replaced with weaker conditions, such as containment up to translation, inequalities of measure, etc.

Progress on the general question of when one convex body must contain a translate (or a congruent copy) of another appears in the work of Gardner and Volčič [13], Groemer [15], Hadwiger [16, 17, 18, 22, 31], Jung [3, 36], Lutwak [26], Rogers [29], Soltan [35], Steinhagen [3, p. 86], Zhou [39, 40], and others (see also [11]). The connection between projections or sections of convex bodies and the comparison of their volumes also lies at the heart of each of two especially notorious inverse problems: the Shephard Problem [34] (solved independently by Petty [28] and Schneider [32]) and the even more difficult Busemann-Petty Problem [5] (see, for example, [1, 4, 8, 9, 12, 14, 23, 24, 25, 27, 37, 38]). A more complete discussion of these and related problems (many of which remain open) can be found in the comprehensive book by Gardner [11].

2. Background

We will require several concepts and established results from convex geometry in Euclidean space. Denote the n-dimensional Euclidean space by $\mathbb{R}^n$, and let $\mathcal{K}_n$ denote the set of compact convex subsets of $\mathbb{R}^n$. The n-dimensional (Euclidean) volume of a set $K \in \mathcal{K}_n$ will be denoted by $V_n(K)$. If u is a unit vector in $\mathbb{R}^n$, denote by K_u the orthogonal projection of a set K onto the subspace $u^\perp$. The boundary of a compact convex set K will be denoted by ∂K.

Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of a compact convex set K; that is,

$$h_K(v) = \max_{x \in K} x \cdot v.$$

The standard separation theorems of convex geometry imply that the support function h_K characterizes the body K; that is, $h_K = h_L$ if and only if $K = L$. If K_i is a sequence in $\mathcal{K}_n$, then $K_i \to K$ in the Hausdorff topology if and only if $h_{K_i} \to h_K$ uniformly when restricted to the unit sphere in $\mathbb{R}^n$.

If u is a unit vector in $\mathbb{R}^n$, denote by K^u the support set of K in the direction of u; that is,

$$K^u = \{x \in K \mid x \cdot u = h_K(u)\}.$$

If P is a convex polytope, then P^u is the face of P having u in its outer normal cone.

Given two compact convex sets $K, L \in \mathcal{K}_n$ and $a, b \geq 0$ denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}.$$

An expression of this form is called a *Minkowski combination* or *Minkowski sum.* Because K, $L \in \mathcal{K}_n$, the set $aK + bL \in \mathcal{K}_n$ as well. Convexity of K also implies that $aK + bK = (a + b)K$ for all $a, b \geq 0$. Support functions are easily shown to satisfy the identity $h_{aK+bL} = ah_K + bh_L$.

The volume of a Minkowski combination satisfies a concavity property called the *Brunn-Minkowski inequality.* Specifically, for $0 \leq t \leq 1$,

$$V_n((1-t)K + tL)^{1/n} \geq (1-t)V_n(K)^{1/n} + tV_n(L)^{1/n}. \tag{2.1}$$

If K and L have interior, then equality holds in (2.1) if and only if K and L are homothetic; that is, iff there exist $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $L = aK + x$. See, for example, any of [3, 10, 33, 36].

The volume $V_n(aK + bL)$ is explicitly given by *Steiner's formula:*

$$V_n(aK + bL) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i V_{n-i,i}(K, L), \tag{2.2}$$

where the *mixed volumes* $V_{n-i,i}(K, L)$ depend only on K and L and the indices i and n. In particular, if we fix two convex sets K and L, then the function $f(a, b) = V_n(aK + bL)$ is a homogeneous polynomial of degree n in the non-negative variables a, b.

Each mixed volume $V_{n-i,i}(K, L)$ is non-negative, continuous in the entries K and L, and monotonic with respect to set inclusion. Note also that $V_{n-i,i}(K, K) = V_n(K)$. If ψ is an affine transformation whose linear component has determinant denoted $\det \psi$, then $V_{n-i,i}(\psi K, \psi L) = |\det \psi|\, V_{n-i,i}(K, L)$. It also follows from (2.2) that $V_{n-i,i}(aK, bL) = a^{n-i} b^i V_{n-i,i}(K, L)$ for all $a, b \geq 0$.

If P is a polytope, then the mixed volume $V_{n-1,1}(P,K)$ satisfies the classical "base-height" formula

$$V_{n-1,1}(P,K) = \frac{1}{n} \sum_{u \perp \partial P} h_K(u) V_{n-1}(P^u), \tag{2.3}$$

where this sum is finite, taken over all outer normals u to the *facets* on the boundary ∂P. These and many other properties of convex bodies and mixed volumes are described in detail in each of [3, 33, 36].

The identity (2.3) implies the following useful containment theorem for simplices, due to Lutwak [26].

Theorem 2.1. *Let $K \in \mathcal{K}_n$ and let $\triangle$ be an n-dimensional simplex. Then $\triangle$ contains a translate of K if and only if*

$$V_{n-1,1}(\triangle, K) \le V_n(\triangle).$$

Proof. Since mixed volumes are translation invariant and monotonic with respect to inclusion of sets, it is immediate that $V_{n-1,1}(\triangle, K) \le V_n(\triangle)$ whenever $\triangle$ contains a translate of K.

Conversely, suppose that $V_{n-1,1}(\triangle, K) \le V_n(\triangle)$. Evidently $\triangle$ contains a translate of K if K is a single point, so let us assume that K is not a single point, so that $V_{n-1,1}(\triangle, K) > 0$. (See, for example, [33, p. 277].)

Let $\alpha > 0$ be maximal such that $\triangle$ contains a translate of αK. Without loss of generality, assume $\alpha K \subseteq \triangle$. If αK does not meet every facet of $\triangle$, then αK can be translated (in the direction of the unit normal to the untouched facet) into the interior of $\triangle$, violating the maximality of α. Therefore, αK meets each facet of $\triangle$, so that

$$h_{\alpha K}(u) = h_\triangle(u)$$

for each unit normal u to facets of $\triangle$. The formula (2.3) now yields

$$V_{n-1,1}(\triangle, \alpha K) = \frac{1}{n} \sum_{u \perp \partial \triangle} h_{\alpha K}(u) V_{n-1}(\triangle^u) = \frac{1}{n} \sum_{u \perp \partial \triangle} h_\triangle(u) V_{n-1}(\triangle^u) = V_n(\triangle),$$

so that

$$\alpha V_{n-1,1}(\triangle, K) = V_{n-1,1}(\triangle, \alpha K) = V_n(\triangle) \ge V_{n-1,1}(\triangle, K).$$

Since $V_{n-1,1}(\triangle, K) > 0$, it follows that $\alpha \ge 1$, so that $\triangle$ contains a translate of K. ☐

Suppose that $\mathscr{F}$ is a family of compact convex sets in $\mathbb{R}^n$. Helly's Theorem [3, 33, 36] asserts that if every $n+1$ sets in $\mathscr{F}$ share a common point, then the entire family shares a common point. In [26] Lutwak used Helly's Theorem to prove the following fundamental criterion for whether a set $L \in \mathcal{K}_n$ contains a translate of another set $K \in \mathcal{K}_n$.

Theorem 2.2 (Lutwak's Containment Theorem)**.** *Let $K, L \in \mathcal{K}_n$. Suppose that, for every n-simplex $\triangle$ such that $L \subseteq \triangle$, there is a vector $v_\triangle \in \mathbb{R}^n$ such that $K + v_\triangle \subseteq \triangle$. Then there is a vector $v \in \mathbb{R}^n$ such that $K + v \subseteq L$.*

A proof of this containment theorem is also given in [22, p. 54]. We will make use of this result in Section 5. Variations of Theorem 2.2 in which circumscribing simplices are replaced with inscribed simplices or circumscribing cylinders are proved in [19] and [21] respectively.

Theorem 2.2 has the following immediate consequence.

Proposition 2.3. *Let $K \in \mathcal{K}_n$. Then $-nK$ contains a translate of K.*

Proof. Suppose that $\triangle$ is an n-dimensional simplex such that $-nK \subseteq \triangle$. It follows that $K \subseteq -\frac{1}{n}\triangle$.

Meanwhile, since $\triangle$ is an n-dimensional simplex, the centroids of its facets are the vertices of a translate of $-\frac{1}{n}\triangle$. It follows that $\triangle$ contains a translate of K. Since this holds for every simplex $\triangle$ that contains $-nK$, it follows from Theorem 2.2 that $-nK$ contains a translate of K. □

3. Interpolating with a simplex

If a convex body K in $\mathbb{R}^n$ has positive volume, then K has at least $n+1$ exposed points [36, p. 89]. It follows from [19, Theorem 2.4] that there exists a simplex $\triangle$ such that every projection $\triangle_u$ contains a translate of the corresponding projection K_u, while $\triangle$ does *not* contain a translate of K.

In general, under these shadow covering conditions, either of the bodies K or $\triangle$ may possibly have larger volume. However, the next theorem asserts that there is always a convex Minkowski combination of K and $\triangle$ whose projections can be translated inside the projections of $\triangle$, while at the same time having *larger volume* than $\triangle$. (See, for example, Figure 1.)

Theorem 3.1. *Suppose that $\triangle$ is an n-simplex, and K is a compact convex set in $\mathbb{R}^n$ such that the following assertions hold:*

(i) Each projection $\triangle_u$ contains a translate of the corresponding projection K_u.

(ii) The simplex $\triangle$ does not contain a translate of K.

Then there exists $t \in (0,1)$ and a convex body $L = (1-t)K + t\triangle$ such that the following assertions hold:

(i)′ Each projection $\triangle_u$ contains a translate of the corresponding projection L_u.

(ii)′ $V_n(L) > V_n(\triangle)$.

Proof. Suppose that $t \in [0,1]$, that $L = (1-t)K + t\triangle$, and that u is a unit vector. We are given in (i) that $\triangle_u$ contains a translate of K_u, so that $K_u + w \subseteq \triangle_u$ for some vector $w \in u^\perp$. It follows that

$$\begin{aligned} L_u + (1-t)w &= (1-t)K_u + t\triangle_u + (1-t)w \\ &= (1-t)(K_u + w) + t\triangle_u \\ &\subseteq (1-t)\triangle_u + t\triangle_u \\ &= \triangle_u, \end{aligned}$$

so that $\triangle_u$ contains a translate of L_u as well. This verifies (i)′ for all $t \in [0,1]$.

Next, we find a value of t so that (ii)′ holds. For $t \in [0,1]$, define

$$f(t) = V_n\Big((1-t)K + t\triangle\Big).$$

Steiner's formula (2.2) implies that f has the polynomial expansion

$$f(t) = \sum_{i=0}^{n} \binom{n}{i} V_{i,n-i}(K,\triangle)(1-t)^i t^{n-i},$$

so that

$$f'(t) = \sum_{i=0}^{n} \binom{n}{i} V_{i,n-i}(K,\triangle)[-i(1-t)^{i-1}t^{n-i} + (n-i)(1-t)^{i}t^{n-i-1}].$$

It follows that

$$f'(1) = nV_{0,n}(K,\triangle) - nV_{1,n-1}(K,\triangle) = nV_n(\triangle) - nV_{1,n-1}(K,\triangle).$$

From the symmetry of mixed volumes, we have $V_{1,n-1}(K,\triangle) = V_{n-1,1}(\triangle,K)$. Since $\triangle$ does not contain a translate of K, Theorem 2.1 now implies that

$$V_{1,n-1}(K,\triangle) = V_{n-1,1}(\triangle,K) > V_n(\triangle),$$

so that $f'(1) < 0$. It follows that $f(t) > f(1)$ for some $t \in (0,1)$. Setting $L = (1-t)K + t\triangle$ for this value of t completes the proof. $\square$

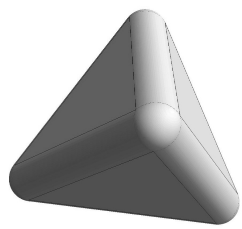

FIGURE 1. A convex Minkowski combination of a regular tetrahedron with a Euclidean ball.

Theorem 3.1, together with [19, Theorem 2.4], implies the following corollary.

Corollary 3.2. *Suppose that $K \in \mathcal{K}_n$ has positive volume. Then there exists a simplex $\triangle$ and $t \in (0,1)$ such that every projection of $\triangle$ contains a translate of the corresponding projection of the body $L = (1-t)\triangle + tK$, while $V_n(L) > V_n(\triangle)$.*

Let $K_0, K_1 \in \mathcal{K}_n$, and suppose that every projection of K_1 contains a translate of the corresponding projection of K_0, while K_1 does not contain K_0. If $t \in (0,1)$, the interpolated body

$$K_t = (1-t)K_0 + tK_1$$

also satisfies these conditions. Theorem 3.1 motivates the following question: Under what conditions on K_0 and K_1 does there exist t so that

$$V_n(K_t) > V_n(K_1)? \tag{3.1}$$

Theorem 3.1 implies there exists such a value t if K_1 is a simplex. On the other hand, there are large classes of convex bodies K_1 for which no such t exists. For example, if K_1 is a centrally symmetric body, then the Brunn-Minkowski inequality (2.1) can be used to show that (3.1) will not hold (see, for example, [20]).

More generally, a set $L \in \mathcal{K}_n$ is called a *cylinder body* if L can be expressed as a limit of Blaschke combinations of cylinders (see [20]). Here a *cylinder* refers to the Minkowski sum in $\mathbb{R}^n$ of an $(n-1)$-dimensional convex body with a line segment.

It turns out that if K_1 is a cylinder body, then no $t \in (0,1)$ will satisfy (3.1), as the next proposition explains.

Proposition 3.3. *Suppose that $K_0, K_1 \in \mathcal{K}_n$ such that every projection of K_1 contains a translate of the corresponding projection of K_0, while K_1 does not contain K_0. If K_1 is an $(n-1)$-cylinder body, then*

$$V_n(K_0) \leq V_n(K_t) \leq V_n(K_1)$$

for all $t \in (0,1)$.

Proof. If K_1 is an $(n-1)$-cylinder body, then

$$V_n(K_0) \leq V_n(K_1) \quad \text{and} \quad V_n(K_t) \leq V_n(K_1),$$

by [20, Theorem 6.1]. It follows from the Brunn-Minkowski inequality (2.1) that, for $t \in (0,1)$,

$$\begin{aligned} V_n(K_t)^{1/n} &= V_n\Big((1-t)K_0 + tK_1\Big)^{1/n} \\ &\geq (1-t)V_n(K_0)^{1/n} + tV_n(K_1)^{1/n} \\ &\geq V_n(K_0)^{1/n}, \end{aligned}$$

so that $V_n(K_0) \leq V_n(K_t)$ as well. □

4. When a body can hide behind a simplex

Denote by Ξ the n-dimensional simplex having vertices at $\{o, e_1, \ldots, e_n\}$, where each e_i is the i-th coordinate unit vector of $\mathbb{R}^n$ and o is the origin. The simplex Ξ has outer facet unit normals given by

$$\{-e_1, -e_2, \ldots, -e_n, v\},$$

where $v \in \mathbb{R}^n$ is the unit vector with coordinates

$$v = \left(\tfrac{1}{\sqrt{n}}, \tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}}\right).$$

Note also that each $\Xi_{e_i} \subseteq \Xi$, since Ξ_{e_i} is the $(n-1)$-dimensional simplex having vertices $\{o, e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$.

Let D denote the cap body formed by the convex hull of Ξ with the point $p \in \mathbb{R}^n$ having coordinates

$$p = \left(\tfrac{1}{n-1}, \tfrac{1}{n-1}, \ldots, \tfrac{1}{n-1}\right).$$

See Figure 2. For each i let $w_i \in \mathbb{R}^n$ denote the vector with coordinates

$$w_i = (1, \ldots, 1, 0, 1 \ldots, 1),$$

where the 0 appears in the i-th coordinate. We can represent D as the intersection of half-spaces

$$D = \left(\bigcap_{i=1}^{n} \{x \mid \ e_i \cdot x \geq 0\}\right) \cap \left(\bigcap_{i=1}^{n} \{x \mid \ w_i \cdot x \leq 1\}\right).$$

For each i, let E_i denote the line segment with endpoints at o and e_i, and let $C_i = \Xi_{e_i} + E_i$ denote a prism (i.e. a cylinder with a simplicial base) that contains Ξ. We can represent each C_i as the intersection of half-spaces

$$C_i = \left(\bigcap_{i=1}^{n} \{x \mid \ e_i \cdot x \geq 0\}\right) \cap \{x \mid \ e_i \cdot x \leq 1\} \cap \{x \mid \ w_i \cdot x \leq 1\}.$$

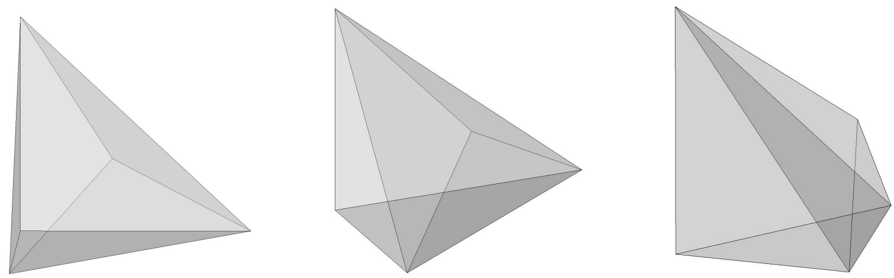

FIGURE 2. Three views of the cap body D (3-dimensional case).

It follows that

$$D = \bigcap_{i=1}^{n} C_i.$$

The next result is fundamental.

Theorem 4.1. *Let $K \in \mathcal{K}_n$, and suppose that, for every unit vector u, the projection Ξ_u contains a translate of the corresponding projection K_u. Then there exists $x \in \mathbb{R}^n$ such that*

$$K + x \subseteq D \subseteq \tfrac{n}{n-1}\Xi.$$

For vectors $v, w \in \mathbb{R}^n$ denote by $v|w^\perp$ the orthogonal projection of the vector v onto the subspace $w^\perp$.

Proof. Translate K so that each coordinate plane $e_i^\perp$ supports K on its positive side. In other words, K is pushed into the corner of the positive orthant of $\mathbb{R}^n$, so that each

$$h_K(-e_i) = 0.$$

Let $y \in K$, with coordinates $y = (y_1, \ldots, y_n)$. The aforementioned repositioning of K implies that each $y_i \geq 0$.

We are given that each projection Ξ_u contains a translate of K_u. In particular, it follows that there exists

$$x = (0, x_2, \ldots, x_n) \in e_1^\perp$$

such that $K_{e_1} + x \subseteq \Xi_{e_1}$. For each $i > 1$,

$$h_{K_{e_1}}(-e_i) + x \cdot (-e_i) = h_{K_{e_1}+x}(-e_i) \leq h_\Xi(-e_i) = 0.$$

Therefore, for each $i > 1$, we have

$$0 \geq h_{K_{e_1}}(-e_i) + x \cdot (-e_i) = h_K(-e_i) + x \cdot (-e_i) = 0 - x_i,$$

so that each $x_i \geq 0$.

Since the coordinates of x are non-negative, we have $x \cdot v \geq 0$, so that

$$h_{K_{e_1}}(v|e_1^\perp) = h_{K_{e_1}}(v) \leq h_{K_{e_1}}(v) + x \cdot v = h_{K_{e_1}+x}(v) \leq h_{\Xi_{e_1}}(v) = h_{\Xi_{e_1}}(v|e_1^\perp),$$

while $h_{K_{e_1}}(-e_i) = 0 = h_{\Xi_{e_1}}(-e_1)$ for each $i > 1$. In other words, K_{e_1} lies inside each of the half-spaces of $e_1^\perp$ that define the simplex Ξ_{e_1}. It follows that

$$K_{e_1} \subseteq \Xi_{e_1},$$

and we can use $x = o$. Moreover, this argument applies in each of the directions $e_1, \ldots, e_n$, so that

$$K_{e_i} \subseteq \Xi_{e_i},$$

for each i.

Since $K_{e_j} \subseteq \Xi_{e_j}$ for each j, the width of K is less than 1 in each coordinate direction e_i. Since $K_{e_i} \subseteq \Xi_{e_i}$ as well, it follows that $K \subseteq C_i$. In other words,

$$K \subseteq \bigcap_{i=1}^{n} C_i = D.$$

The vertices of D are $\{o, e_1, \ldots, e_n, p\}$. Evidently, $o, e_1, \ldots, e_n \in \Xi \subseteq \frac{n}{n-1}\Xi$. Since p has positive coordinates which sum to $\frac{n}{n-1}$, we have $p \in \frac{n}{n-1}\Xi$ as well. It follows that $D \subseteq \frac{n}{n-1}\Xi$. □

It immediately follows from Theorem 4.1 that, if every projection Ξ_u contains a translate of the corresponding projection K_u, then

$$V_n(K) \leq \left(\tfrac{n}{n-1}\right)^n V_n(\Xi). \tag{4.1}$$

Since $\left(\frac{n}{n-1}\right)^n$ decreases to e as $n \to \infty$, this gives a universal upper bound on the ratio

$$\frac{V_n(K)}{V_n(\Xi)}$$

under the condition of covering projections.

The next proposition will allow us to generalize these observations from the special case of covering by the simplex Ξ to covering by an arbitrary n-simplex.

Proposition 4.2. *Let $K, L \in \mathcal{K}_n$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear transformation. Then L_u contains a translate of K_u for all unit directions u if and only if $(\psi L)_u$ contains a translate of $(\psi K)_u$ for all u.*

This proposition implies that nothing is gained (or lost) by allowing more general (possibly non-orthogonal) linear projections.

Proof. For $S \subseteq \mathbb{R}^n$ and a non-zero vector u, let $\mathcal{L}_S(u)$ denote the set of straight lines in $\mathbb{R}^n$ parallel to u and meeting the set S. The projection L_u contains a translate K_u for each unit vector u if and only if, for each u, there exists v_u such that

$$\mathcal{L}_{K+v_u}(u) \subseteq \mathcal{L}_L(u). \tag{4.2}$$

But $\mathcal{L}_{K+v_u}(u) = \mathcal{L}_K(u) + v_u$ and $\psi\mathcal{L}_K(u) = \mathcal{L}_{\psi K}(\psi u)$. It follows that (4.2) holds if and only if $\mathcal{L}_K(u) + v_u \subseteq \mathcal{L}_L(u)$, which, in turn, holds if and only if

$$\mathcal{L}_{\psi K}(\psi u) + \psi v_u \subseteq \mathcal{L}_{\psi L}(\psi u) \quad \text{for all units } u.$$

Set

$$\tilde{u} = \frac{\psi u}{|\psi u|} \quad \text{and} \quad \tilde{v} = \psi v_u.$$

The relation (4.2) now holds if and only if, for each $\tilde{u}$, there exists $\tilde{v}$ such that

$$\mathcal{L}_{\psi K}(\tilde{u}) + \tilde{v} \subseteq \mathcal{L}_{\psi L}(\tilde{u}),$$

which holds if and only if $(\psi L)_{\tilde{u}}$ contains a translate of $(\psi K)_{\tilde{u}}$ for all $\tilde{u}$. □

Proposition 4.2 implies that the projection covering relation is preserved by invertible affine transformations. Since every n-dimensional simplex can be expressed as the affine image of the simplex Ξ, the volume inequality (4.1) continues to hold when Ξ is replaced by *any* simplex whose projections cover those of K. In the next section we will generalize this observation still further to an even larger class of sets. However, the volume bound (4.1) can also be strengthened for the special case in which the projections of K are covered by those of a simplex.

Theorem 4.3. *Let $K \in \mathcal{K}_n$ and let T denote an n-dimensional simplex such that, for every unit vector u, the projection T_u contains a translate of the corresponding projection K_u. Then*

$$V_n(K) \leq \tfrac{n}{n-1} V_n(T).$$

Proof. Without loss of generality (scaling as needed) we may assume that $V_n(T) = V_n(\Xi)$, where Ξ is the special simplex defined at the beginning of this section. Applying volume-preserving affine transformations as needed, Proposition 4.2 implies that we may also assume, without loss of generality, that $T = \Xi$. In this case the proof of Theorem 4.1 implies that K lies inside the cap body D. An elementary computation shows that

$$V_n(D) = \tfrac{n}{n-1} V_n(\Xi) = \tfrac{n}{n-1} V_n(T).$$

The theorem now follows from the monotonicity of volume. □

5. When one body can hide behind another

We now re-state and prove the Main Theorem of this article, which generalizes some of the results of the previous section to the case of *any* two compact convex sets K, L in $\mathbb{R}^n$ such that orthogonal projections of L contain translates of the corresponding projections of K.

Theorem 5.1. *Let $K, L \in \mathcal{K}_n$. Suppose that, for every unit vector u, the projection L_u contains a translate of the corresponding projection K_u. Then there exists $x \in \mathbb{R}^n$ such that*

$$K + x \subseteq \tfrac{n}{n-1} L.$$

This theorem gives a sharp bound for containment by sets with covering projections. To see this, recall that if $K \in \mathcal{K}_n$, the set $-nK$ contains a translate of K, by Proposition 2.3. Then consider the case in which K is the regular unit edge n-simplex $\triangle$, and $L = (n-1)(-\triangle)$. For each direction u, the projection $-(n-1)\triangle_u$ contains a translate of $\triangle_u$. Meanwhile, the smallest dilate of $-\triangle$ to contain a translate of $K = \triangle$ is

$$n(-\triangle) = \tfrac{n}{n-1}(n-1)(-\triangle) = \tfrac{n}{n-1} L.$$

It follows that the coefficient $\frac{n}{n-1}$ in Theorem 5.1 cannot be improved.

Proof of Theorem 5.1. Let T be any n-simplex that contains L. Since each projection L_u contains a translate of K_u, it follows that each T_u contains a translate of K_u. Let Ξ and D again denote the simplex and cap body defined in the previous section. Let ψ be an invertible affine transformation ψ such that $\psi T = \Xi$. By Proposition 4.2 each projection Ξ_u contains a translate of the corresponding projection $(\psi K)_u$ of the body ψK. By Theorem 4.1 there is $x \in \mathbb{R}^n$ such that

$$\psi K + x \subseteq D \subseteq \tfrac{n}{n-1} \Xi.$$

Since ψ is affine and invertible, it follows that the simplex

$$\tilde{T} = \tfrac{n}{n-1} T$$

contains a translate of K. Meanwhile $\tilde{T}$ circumscribes $\frac{n}{n-1}L$ if and only if T circumscribes L. So we have shown that every circumscribing simplex $\tilde{T}$ of $\frac{n}{n-1}L$ contains a translate of K. It follows from the Lutwak Containment Theorem 2.2 that $\frac{n}{n-1}L$ contains a translate of K. □

Corollary 5.2. *Let K, $L \in \mathcal{K}_n$. Suppose that, for every unit vector u, the projection L_u contains a translate of the corresponding projection K_u. Then*

$$V_n(K) \leq \left(\tfrac{n}{n-1}\right)^n V_n(L).$$

Proof. By Theorem 5.1 there exists $x \in \mathbb{R}^n$ such that

$$K + x \subseteq \tfrac{n}{n-1} L,$$

so that

$$V_n(K) = V_n(K+x) \leq V_n\left(\tfrac{n}{n-1}L\right) = \left(\tfrac{n}{n-1}\right)^n V_n(L).$$

□

In particular, if each L_u contains a translate of K_u, then there is a constant $c_n \in \mathbb{R}$ independent of $K, L \in \mathcal{K}_n$ such that

$$V_n(K) \leq c_n V_n(L), \tag{5.1}$$

where $c_n \to e$ as $n \to \infty$.

The volume ratio bound (5.1) gives a substantial improvement over those previously known. In [20] circumscribing cylinders were used to show that $c_n \leq n$ for all $n \geq 1$. A simple argument also implies that $c_2 = 3/2$ is the best possible result in dimension 2. More generally, an upper bound for c_n can also be obtained using the Rogers-Shephard inequality (also known as the *difference body inequality* [6, 30], [33, p. 409]). This inequality asserts that, for $K \in \mathcal{K}_n$,

$$V_n(K + (-K)) \leq \binom{2n}{n} V_n(K). \tag{5.2}$$

If $V_n(K) > 0$, then equality holds in (5.2) if and only if K is a simplex.

To obtain a bound for c_n using (5.2), suppose $K, L \in \mathcal{K}_n$ and that each L_u contains a translate of K_u. It follows that the same relation holds for the Minkowski symmetrals $\frac{1}{2}(L + (-L))$ and $\frac{1}{2}(K + (-K))$. Since these symmetrals are both *centrally symmetric* it follows that

$$\tfrac{1}{2}(K + (-K)) \subseteq \tfrac{1}{2}(L + (-L)),$$

so that

$$V_n(K) \leq V_n\left(\tfrac{1}{2}(K + (-K))\right) \leq V_n\left(\tfrac{1}{2}(L + (-L))\right) \leq \frac{1}{2^n}\binom{2n}{n} V_n(L),$$

where the first inequality follows from the Brunn-Minkowski inequality (2.1) and the final inequality is the Rogers-Shephard inequality (5.2). In dimension 2 this yields the sharp bound $c_2 = 3/2$, where equality is attained when L is a triangle and $K = \frac{1}{2}(L + (-L))$.

However, for dimensions 3 and above, the Rogers-Shephard inequality no longer gives the best possible bound for c_n. More complicated results of Ball [2] (see also [11, pp. 163-164]) imply that if the area of each projection of L exceeds the corresponding area of each projection of K (a much weaker assumption than actual covering of projections), then c_n grows with order at most $\sqrt{n}$, with a universal (weak) bound of

$$V_n(K) \leq 1.1696\sqrt{n}V_n(L). \tag{5.3}$$

The bound (5.3) implies that $c_3 \leq 2.026$, whereas the Rogers-Shephard bound only tells us that $c_3 \leq 2.5$. The inequality (5.3) still gives the best known bound for c_n when $3 \leq n \leq 6$, although all known numerical evidence suggests these bounds can be substantially improved (see also Section 7). Moreover, the bound (5.3) increases without limit as $n \to \infty$.

Theorem 5.1 (and Corollary 5.2) implies that if projections of L can cover projections of K, then c_n is actually bounded by a ***universal constant*** independent of the dimension n. The inequality (5.3) implies that

$$c_n \leq 1.1696\sqrt{6} \approx 2.865,$$

for $n \leq 6$, but gives bounds larger than 3 and increasing without limit for $n \geq 7$. Meanwhile, Corollary 5.2 implies that

$$c_n \leq \left(\tfrac{n}{n-1}\right)^n \leq \left(\tfrac{7}{6}\right)^7 \approx 2.942$$

for $n \geq 7$, giving a universal volume ratio bound of $c_n \leq 2.942$ in all finite dimensions. Possible improvements for this universal bound are discussed in Section 7.

6. Projections to intermediate dimensions

Theorem 5.1 generalizes easily to projections onto an arbitrary lower dimension. In order to obtain similar asymptotic bounds (as the ambient dimension $n \to \infty$), these analogous results are best expressed in terms of the *co-dimension* of the projections.

If ξ is a subspace of $\mathbb{R}^n$ and $K \in \mathcal{K}_n$, we will denote by K_ξ the orthogonal projection of K onto ξ.

Theorem 6.1. *Let K, $L \in \mathcal{K}_n$, and let $d \in \{1, \ldots, n-1\}$. Suppose that, for every $(n-d)$-dimensional subspace $\xi \subseteq \mathbb{R}^n$, the projection L_ξ contains a translate of the corresponding projection K_ξ. Then there exists $x \in \mathbb{R}^n$ such that*

$$K + x \subseteq \tfrac{n}{n-d}L.$$

This theorem gives a sharp bound for containment by sets with covering projections. To see this, recall that if $K \in \mathcal{K}_n$, then each $(n-d)$-dimensional projection $(n-d)(-K)_\xi$ of $(n-d)(-K)$ contains a translate of the corresponding projection K_ξ, by Proposition 2.3. Then consider the case in which K is the regular unit edge n-simplex $\triangle$, and $L = (n-d)(-\triangle)$. For each $(n-d)$-dimensional subspace ξ, the projection $(n-d)(-\triangle)_\xi$ contains a translate of $\triangle_\xi$. Meanwhile, the smallest dilate of $-\triangle$ to contain a translate of $K = \triangle$ is

$$n(-\triangle) = \tfrac{n}{n-d}(n-d)(-\triangle) = \tfrac{n}{n-d}L.$$

It follows that the coefficient $\frac{n}{n-d}$ in Theorem 6.1 cannot be improved.

Proof of Theorem 6.1. The case of $d = 1$ is addressed by Theorem 5.1. If $d > 1$ let $u \in \xi^{\perp}$ be a unit vector, and let $\overline{u}$ denote the line through the origin spanned by u. By Theorem 5.1, applied within the $(n-d+1)$-dimensional space $\xi \oplus \overline{u}$, there is a vector y such that

$$K_{\xi\oplus\overline{u}} + y \subseteq \tfrac{n-d+1}{n-d} L_{\xi\oplus\overline{u}}.$$

In other words, for every $(n-d+1)$-dimensional subspace $\xi' \subseteq \mathbb{R}^n$ the set $\frac{n-d+1}{n-d}L_{\xi'}$ contains a translate of $K_{\xi'}$. After d iterations of this argument we obtain a vector x such that

$$K + x \subseteq \tfrac{n}{n-1} \cdots \tfrac{n-d+2}{n-d+1}\tfrac{n-d+1}{n-d} L = \tfrac{n}{n-d} L.$$

□

Corollary 6.2. *Let K, $L \in \mathcal{K}_n$, and let $d \in \{1, \ldots, n-1\}$. Suppose that, for every $(n-d)$-dimensional subspace $\xi \subseteq \mathbb{R}^n$, the projection L_ξ contains a translate of the corresponding projection K_ξ. Then*

$$V_n(K) \leq \left(\tfrac{n}{n-d}\right)^n V_n(L).$$

Proof. By Theorem 6.1 there exists $x \in \mathbb{R}^n$ such that

$$K + x \subseteq \tfrac{n}{n-d} L,$$

so that

$$V_n(K) = V_n(K+x) \leq V_n\left(\tfrac{n}{n-d}L\right) = \left(\tfrac{n}{n-d}\right)^n V_n(L).$$

□

Note that, after fixing the co-dimension d, we have

$$\lim_{n\to\infty}\left(\tfrac{n}{n-d}\right)^n = \lim_{n\to\infty}\left(1+\tfrac{d}{n-d}\right)^n = \lim_{n\to\infty}\left(1+\tfrac{d}{n-d}\right)^d\left(1+\tfrac{d}{n-d}\right)^{n-d} = e^d.$$

Corollary 6.2 implies that, if L_ξ contains a translate of K_ξ for every $(n-d)$-dimensional subspace $\xi \subseteq \mathbb{R}^n$, then there is a constant $c_{n,d} \in \mathbb{R}$ independent of $K, L \in \mathcal{K}_n$ such that

$$V_n(K) \leq c_{n,d} V_n(L), \tag{6.1}$$

where $c_{n,d} \to e^d$ as $n \to \infty$. It follows that, for fixed co-dimension d, the coefficient $c_{n,d}$ can be replaced by a universal constant γ_d independent of the bodies K and L and independent of the ambient dimension n.

7. Concluding remarks and open questions

Numerical evidence suggests that the volume ratio bounds in this article can almost certainly be improved [7]. In Section 5 we showed that if each projection L_u contains a translate of K_u, then

$$V_n(K) \leq c V_n(L), \tag{7.1}$$

where c is a constant independent of the dimension n, and where $c < 2.942$. However, computational evidence suggests that c is much smaller.

If we fix the dimension n, then Theorem 4.3 gives a value of $c = \frac{n}{n-1}$ when the body L is an n-simplex. On the other hand, previous work [20] implies that

$$V_n(K) \leq V_n(L),$$

in the special case where L is a cylinder, or even a cylinder body (that is, a limit of Blaschke combinations of cylinders). This suggests that simplices may be the worst case scenario for bodies with covering shadows, and motivates the following conjecture:

Conjecture. *Let $K, L \in \mathcal{K}_n$ and suppose that L_u contains a translate of K_u for every unit vector $u \in \mathbb{R}^n$. Then*

$$V_n(K) \leq \frac{n}{n-1} V_n(L). \tag{7.2}$$

This conjecture is already known to be true in dimension 2 (indeed, we observed in Section 5 that $c_2 = 3/2$ is the best possible bound), but remains open for dimensions $n \geq 3$. If this conjecture is true, then the universal volume ratio constant for all dimensions $n \geq 2$ would satisfy $c = 3/2$.

Even if the conjecture above is proven correct, it remains to determine the best upper bound for the ratio

$$c_n = \frac{V_n(K)}{V_n(L)}$$

in each dimension separately, for the conjectured bound (7.2) does not appear to be sharp in dimensions $n \geq 3$. Examples investigated so far suggest that the highest volume ratio ought to occur when a suitable convex Minkowski combination of a simplex $\triangle$ with the scaled reflection $(n-1)(-\triangle)$ hides behind the set $(n-1)(-\triangle)$.

A direct computation [7] shows that if $\triangle$ is a tetrahedron in $\mathbb{R}^3$, and if

$$K = \left(1 - \frac{1+\sqrt{56}}{11}\right)\triangle + \left(\frac{1+\sqrt{56}}{11}\right)(-2\triangle) \quad \text{and} \quad L = -2\triangle, \tag{7.3}$$

then each projection L_u contains a translate of K_u (by Proposition 2.3, applied in dimension 2), while

$$\frac{V_n(K)}{V_n(L)} \approx 1.1634.$$

See Figure 3. We conjecture that the best possible bound for the volume ratio in

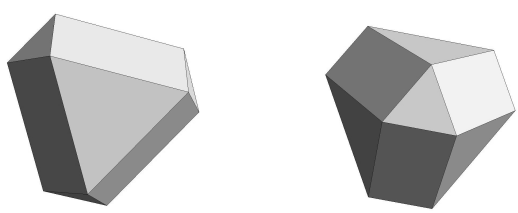

FIGURE 3. Two views of the Minkowski combination K of a regular tetrahedron with its reflection, as specified in (7.3).

dimension 3 is the value $c_3 \approx 1.1634$ occurring with the pair of bodies specified in (7.3), and that analogous computations with simplices in $\mathbb{R}^n$ will yield the best bounds for c_n. However, these assertions remain conjectures at this point.

Open Question 1. What is the best possible value for the volume ratio bound c_n for each particular dimension $n \geq 3$? For $c_{n,d}$?

Open Question 2. What is the best possible value for the universal volume ratio bound c for all dimensions $n \geq 3$? For γ_d?

Open Question 3. Let $K, L \in \mathcal{K}_n$, and let $d \in \{1, \ldots, n-1\}$. Suppose that, for each $(n-d)$-dimensional subspace ξ of $\mathbb{R}^n$, the orthogonal projection L_ξ of K contains a translate of K_ξ.

Under what simple (easy to state, easy to verify) additional conditions does it follow that $V_n(K) \leq V_n(L)$?

Some partial answers to the third question are given in [20]. There it is shown that if K_ξ can be translated inside L_ξ for all $(n-d)$-dimensional subspaces ξ, then K has smaller volume than L whenever L can be approximated by Blaschke combinations of $(n-d)$-decomposable sets. Moreover, in [21] it is shown that, for example, if projections of a right square pyramid Q contain translates of the projections of a convex body K in $\mathbb{R}^3$, then Q contains a translate of K (and so certainly has greater volume). Since Q does not appear to be a cylinder body (a class of bodies not yet easily characterized), it is likely that a larger class of examples exist for bodies L whose volume exceeds that of any body K having smaller shadows (up to translation).

The question of volume comparison was originally motivated by the following.

Open Question 4. Let $K, L \in \mathcal{K}_n$, and let $d \in \{1, \ldots, n-1\}$. Suppose that, for each $(n-d)$-dimensional subspace ξ of $\mathbb{R}^n$, the orthogonal projection L_ξ of K contains a translate of K_ξ.

Under what simple (easy to state, easy to verify) additional conditions does it follow that L contains a translate of K?

Some partial answers to this question are given in [19] and [21].

All of these many questions can be re-phrased allowing for (specified subgroups of) rotations (and reflections) as well as translations. However, the results obtained so far rely on the observation that the set of translates of K that fit inside L, that is, the set

$$\{v \in \mathbb{R}^n \mid K + v \subseteq L\},$$

is itself a compact convex set in $\mathbb{R}^n$. By contrast, the set of rigid motions of K that fit inside L will lie in a more complicated Lie group. For this reason (at least) the questions of covering via rigid motions may be more difficult to address than the case in which only translation is allowed.

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References

1. K. Ball, *Some remarks on the geometry of convex sets*, Geometric Aspects of Functional Analysis (J. Lindenstrauss and V.D. Milman, eds.), Springer Lecture Notes in Mathematics, vol. 1317, Springer Verlag, Berlin, 1988. MR950983 (89h:52009)
2. K. Ball, *Shadows of convex bodies*, Trans. Amer. Math. Soc. **327** (1991), 891–901. MR1035998 (92a:52011)

3. T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987. MR920366 (88j:52001)
4. J. Bourgain, *On the Busemann-Petty problem for perturbations of the ball*, Geom. Funct. Anal. **1** (1991), 1–13. MR1091609 (92c:52008)
5. H. Busemann and C. M. Petty, *Problems on convex bodies*, Math. Scand. **4** (1956), 88–94. MR0084791 (18:922b)
6. G. D. Chakerian, *Inequalities for the difference body of a convex body*, Proc. Amer. Math. Soc. **18** (1967), 879–884. MR0218972 (36:2055)
7. C. Chen, *Maximizing volume ratios for shadow covering by tetrahedra*, arXiv:1201.2580v1 (2012).
8. R. J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435–445. MR1201126 (94e:52008)
9. R. J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. Math. (2) **140** (1994), 435–447. MR1298719 (95i:52005)
10. R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405. MR1898210 (2003f:26035)
11. R. J. Gardner, *Geometric Tomography (2nd Ed.)*, Cambridge University Press, New York, 2006. MR2251886 (2007i:52010)
12. R. J. Gardner, A. Koldobsky, and T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. Math. (2) **149** (1999), 691–703. MR1689343 (2001b:52011)
13. R. J. Gardner and A. Volčič, *Convex bodies with similar projections*, Proc. Amer. Math. Soc. **121** (1994), 563–568. MR1185262 (94h:52005)
14. A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239–244. MR1099772 (92c:52009)
15. H. Groemer, *Ein Satz über konvexe Körper und deren Projektionen*, Portugal. Math. **21** (1962), 41–43. MR0142055 (25:5449)
16. H. Hadwiger, *Gegenseitige Bedeckbarkeit zweier Eibereiche und Isoperimetrie*, Vierteljschr. Naturforsch. Gesellsch. Zürich **86** (1941), 152–156. MR0007274 (4:112c)
17. H. Hadwiger, *Überdeckung ebener Bereiche durch Kreise und Quadrate*, Comment. Math. Helv. **13** (1941), 195–200. MR0004995 (3:90f)
18. H. Hadwiger, *Seitenrisse konvexer Körper und Homothetie*, Elem. Math. **18** (1963), 97–98. MR0155232 (27:5168)
19. D. Klain, *Containment and inscribed simplices*, Indiana Univ. Math. J. **59** (2010), 1231–1244. MR2815032
20. D. Klain, *Covering shadows with a smaller volume*, Adv. Math. **224** (2010), 601–619. MR2609017 (2011d:52012)
21. D. Klain, *If you can hide behind it, can you hide inside it?*, Trans. Amer. Math. Soc. **363** (2011), 4585–4601. MR2806685
22. D. Klain and G.-C. Rota, *Introduction to Geometric Probability*, Cambridge University Press, New York, 1997. MR1608265 (2001f:52009)
23. A. Koldobsky, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), 827–840. MR1637955 (99i:52005)
24. D. G. Larman and C. A. Rogers, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika **22** (1975), 164–175. MR0390914 (52:11737)
25. E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261. MR963487 (90a:52023)
26. E. Lutwak, *Containment and circumscribing simplices*, Discrete Comput. Geom. **19** (1998), 229–235. MR1600051 (99a:52009)
27. A. Papadimitrakis, *On the Busemann-Petty problem about convex centrally symmetric bodies in* $\mathbb{R}^n$, Mathematika **39** (1992), 258–266. MR1203282 (94a:52019)
28. C. M. Petty, *Projection bodies*, Proceedings, Coll. Convexity, Copenhagen, 1965, vol. Københavns Univ. Mat. Inst., 1967, pp. 234–241. MR0216369 (35:7203)
29. C. A. Rogers, *Sections and projections of convex bodies*, Portugal. Math. **24** (1965), 99–103. MR0198344 (33:6502)
30. C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Arch. Math. **8** (1957), 220–233. MR0092172 (19:1073f)

31. L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, MA, 1976. MR0433364 (55:6340)
32. R. Schneider, *Zur einem Problem von Shephard über die Projektionen konvexer Körper.*, Math Z. **101** (1967), 71–82. MR0218976 (36:2059)
33. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, New York, 1993. MR1216521 (94d:52007)
34. G. C. Shephard, *Shadow systems of convex bodies*, Israel J. Math. **2** (1964), 229–236. MR0179686 (31:3931)
35. V. Soltan, *Convex sets with homothetic projections*, Beiträge Algebra Geom. **51** (2010), 237–249. MR2650489 (2011c:52010)
36. R. Webster, *Convexity*, Oxford University Press, New York, 1994. MR1443208 (98h:52001)
37. G. Zhang, *Intersection bodies and the Busemann-Petty inequalities in* $\mathbb{R}^4$, Ann. Math. (2) **140** (1994), no. 2, 331–346. MR1298716 (95i:52004)
38. G. Zhang, *A positive solution to the Busemann-Petty problem in* $\mathbb{R}^4$, Ann. Math. (2) **149** (1999), 535–543. MR1689339 (2001b:52010)
39. J. Zhou, *The sufficient condition for a convex body to contain another in* $\mathbb{R}^4$, Proc. Amer. Math. Soc. **121** (1994), 907–913. MR1184090 (94i:52007)
40. J. Zhou, *Sufficient conditions for one domain to contain another in a space of constant curvature*, Proc. Amer. Math. Soc. **126** (1998), 2797–2803. MR1451838 (98k:52016)

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