$$
\begin{aligned}
&= T_f(x^{g(k)}) + T_f(x^{g(\infty)}) \quad \text{(since } T_f \text{ is linear)} \\
&= (x^{f(g(k))} + x^{f(\infty)}) + (x^{f(g(\infty))} + x^{f(\infty)}) \\
&= x^{f(g(k))} + x^{f(g(\infty))} \quad \text{(since } u + u = 0 \text{ for all } u \in \mathbb{F}_8) \\
&= T_{f \circ g}(x^k).
\end{aligned}
$$

3. *Proof that T is a homomorphism mapping into* $\mathrm{GL}(\mathbb{F}_8)$. We have seen that each element of SLF(7) can be written as a product of the generators r, t, and δ (using only positive powers, in fact). Since $T(r)$, $T(t)$, and $T(\delta)$ are known to lie in $\mathrm{GL}(\mathbb{F}_8)$, repeated application of the lemma shows that $T(h)$ lies in $\mathrm{GL}(\mathbb{F}_8)$ for *all* $h \in$ SLF(7). Having drawn this conclusion, the lemma now shows that T is a group homomorphism.
4. *Proof that T is a bijection.* So far, we know that T is a group homomorphism mapping SLF(7) *into* $\mathrm{GL}(\mathbb{F}_8)$. T is actually *onto*, since the image of T contains $\langle T(r), T(t), T(\delta)\rangle$, which is the whole group $\mathrm{GL}(\mathbb{F}_8)$. Since SLF(7) and $\mathrm{GL}(\mathbb{F}_8)$ both have 168 elements, T must also be one-to-one.

Our proof that $\mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, 2)$ is now complete. We leave it as a challenge for the reader to find an explicit description of the inverse bijection $T^{-1} : \mathrm{GL}(\mathbb{F}_8) \to$ SLF(7).

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Angles as Probabilities

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Almost everyone knows that the inner angles of a triangle sum to 180°. But if you ask the typical mathematician how to sum the solid inner angles over the vertices of a tetrahedron, you are likely to receive a blank stare or a mystified shrug. In some cases you may be directed to the Gram-Euler relations for higher-dimensional polytopes [**4, 5, 7, 8**], a 19th-century result unjustly consigned to relative obscurity. But the answer is really much simpler than that, and here it is:

> The sum of the solid inner vertex angles of a tetrahedron T, divided by 2π, gives the probability that the orthogonal projection of T onto a random 2-plane is a triangle.

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How simple is that? We will prove a more general theorem (Theorem 1) for simplices in $\mathbb{R}^n$, but first consider the analogous assertion in $\mathbb{R}^2$. The sum in radians of the angles of a triangle (2-simplex) T, when divided by the length π of the unit semicircle, gives the probability that the orthogonal projection of T onto a random line is a convex segment (1-simplex). Since this is *always* the case, the probability is equal to 1, and the inner angle sum for every triangle is the same. By contrast, a higher-dimensional n-simplex may project one dimension down either to an $(n-1)$-simplex or to a lower-dimensional convex polytope having $n+1$ vertices. The inner angle sum gives a measure of how often each of these possibilities occurs.

Let us make the notion of "inner angle" more precise. Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ centered at the origin. Recall that $\mathbb{S}^{n-1}$ has $(n-1)$-dimensional volume (i.e., surface area) $n\omega_n$, where $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$ is the Euclidean volume of the unit ball in $\mathbb{R}^n$.

Suppose that P is a convex polytope in $\mathbb{R}^n$, and let v be any point of P. The solid inner angle $a_P(v)$ of P at v is given by

$$a_P(v) = \{u \in \mathbb{S}^{n-1} \mid v + \epsilon u \in P \quad \text{for some } \epsilon > 0\}.$$

Let $\alpha_P(v)$ denote the measure of the solid angle $a_P(v) \subseteq \mathbb{S}^{n-1}$, given by the usual surface area measure on subsets of the sphere. For the moment we are primarily concerned with values of $\alpha_P(v)$ when v is a *vertex* of P.

If u is a unit vector, then denote by P_u the orthogonal projection of P onto the subspace $u^\perp$ in $\mathbb{R}^n$. Let v be a vertex of P. The projection v_u lies in the relative interior of P_u if and only if there exists $\epsilon \in (-1, 1)$ such that $v + \epsilon u$ lies in the interior of P. This holds if and only if u lies in the interior of $\pm a_P(v)$. If u is a random unit vector in $\mathbb{S}^{n-1}$, then

$$\texttt{Probability}[v_u \in \text{relative interior}(P_u)] = \frac{2\alpha_P(v)}{n\omega_n}. \tag{1}$$

This gives the probability that v_u is no longer a vertex of P_u.

For *simplices* we now obtain the following theorem.

Theorem 1 (Simplicial Angle Sums). *Let Δ be an n-simplex in $\mathbb{R}^n$, and let u be a random unit vector. Denote by p_Δ the probability that the orthogonal projection Δ_u is an $(n-1)$-simplex. Then*

$$p_\Delta = \frac{2}{n\omega_n} \sum_v \alpha_\Delta(v), \tag{2}$$

where the sum is taken over all vertices of the simplex Δ.

Proof. Since Δ is an n-simplex, Δ has $n+1$ vertices, and a projection Δ_u has either n or $n+1$ vertices. (Since Δ_u spans an affine space of dimension $n-1$, it cannot have fewer than n vertices.) In other words, either exactly 1 vertex of Δ projects to the relative interior of Δ_u, so that Δ_u is an $(n-1)$-simplex, or none of them do. By the law of alternatives, the probability p_Δ is now given by the sum of the probabilities (1), taken over all vertices of the simplex Δ. ■

The probability (2) is always equal to 1 for a 2-dimensional simplex (i.e., any triangle). The *regular* tetrahedron in $\mathbb{R}^3$ has solid inner angle measure

$$\alpha_T(v) = 3\arccos(1/3) - \pi$$

at each of its 4 vertices, so that (2) yields $p_\Delta \approx 0.351$. For more general 3-simplices (tetrahedra) the probability may take *any* value $0 < p_\Delta < 1$. To obtain a value of p_Δ close to 1 for a tetrahedron, consider the convex hull of an equilateral triangle in $\mathbb{R}^3$ with a point outside the triangle, but very close to its center. To obtain p_Δ close to 0, consider the convex hull of two skew line segments in $\mathbb{R}^3$ whose centers are very close together (forming a tetrahedron that is almost a parallelogram). Similarly, for $n \geq 3$ the solid vertex angle sum of an n-simplex varies within a range

$$0 < \sum_v \alpha_T(v) < \frac{n\omega_n}{2}.$$

Equality at either end is obtained only if one allows for the degenerate limiting cases. These bounds were obtained earlier by Gaddum [**2, 3**] and Barnette [**1**], using more complicated methods.

Similar considerations apply to the solid angles at arbitrary faces of convex polytopes. Suppose that F is a k-dimensional face of a convex polytope P, for some k with $0 \leq k \leq \dim P$. The solid inner angle measure $\alpha_P(x)$ is the same at every point x in the relative interior of F. Denote this value by $\alpha_P(F)$. In analogy to (1), for any x in the relative interior of F, we have

$$\texttt{Probability}[x_u \in \text{relative interior}(P_u)] = \frac{2\alpha_P(F)}{n\omega_n}. \tag{3}$$

Omitting cases of measures zero, this gives the probability that a proper face F is no longer a face of P_u. (Note that $\dim F_u = \dim F$ for all directions u except a set of measure zero.) Taking complements, we have

$$\texttt{Probability}[F_u \text{ is a proper face of } P_u] = 1 - \frac{2\alpha_P(F)}{n\omega_n}. \tag{4}$$

For $0 \leq k \leq n-1$, denote by $f_k(P)$ the number of k-dimensional faces of a polytope P. The sum of the probabilities (4) gives the expected number of k-faces of the projection of P onto a random hyperplane $u^\perp$; that is,

$$\texttt{Exp}[f_k(P_u)] = \sum_{\dim F = k} \left(1 - \frac{2\alpha_P(F)}{n\omega_n}\right) = f_k(P) - \frac{2}{n\omega_n} \sum_{\dim F = k} \alpha_P(F), \tag{5}$$

where the sums are taken over k-faces F of the polytope P.

If P is a convex polygon in $\mathbb{R}^2$, then P_u is always a line segment with exactly 2 vertices, that is, $f_0(P_u) = 2$. In this case the expectation identity (5) yields the familiar

$$\sum_v \alpha_P(v) = \pi(f_0(P) - 2).$$

If P is a convex polytope in $\mathbb{R}^3$, then P_u is a convex polygon, which always has exactly as many vertices as edges; that is, $f_0(P_u) = f_1(P_u)$. Therefore $\texttt{Exp}[f_0(P_u)] = \texttt{Exp}[f_1(P_u)]$, and the expectation identities (5) imply that

$$\frac{1}{2\pi} \sum_{\text{vertices } v} \alpha_P(v) - \frac{1}{2\pi} \sum_{\text{edges } e} \alpha_P(e) = f_0(P) - f_1(P) = 2 - f_2(P),$$

where the second equality follows from the classical Euler formula $f_0 - f_1 + f_2 = 2$ for convex polyhedra in $\mathbb{R}^3$.

These arguments were generalized by Perles and Shephard [**7**] (see also [**1**], [**4**, p. 315a], [**9**]) to give a simple proof of the classical Gram-Euler identity for convex polytopes:

$$\sum_{F \subseteq \partial P} (-1)^{\dim F} \alpha_P(F) = (-1)^{n-1} n\omega_n, \tag{6}$$

where the sum is taken over all proper faces F of an n-dimensional convex polytope P. In the general case one applies the additivity of expectation to alternating sums over k of the identities (5), obtaining identities that relate the Euler numbers of the boundaries of P and P_u. Since the boundary of P is a piecewise-linear $(n-1)$-sphere, while the boundary of P_u is a piecewise-linear $(n-2)$-sphere, these Euler numbers are easily computed, and (6) follows.

The Gram-Euler identity (6) can be viewed as a discrete analogue of the Gauss-Bonnet theorem, and has been generalized to Euler-type identities for angle sums over polytopes in spherical and hyperbolic spaces [**4, 5, 8**], as well as for mixed volumes and other valuations on polytopes [**6**].

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Jump Home and Shift: An Acyclic Operation on Permutations

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Let $n \geq 1$ be fixed, and denote by S_n the set of all permutations of $[n] = \{1, \dots, n\}$. We write a permutation π as a vector $\pi = (\pi(1), \pi(2), \dots, \pi(n))$, and we say that the element $\pi(i)$ is in the ith position. We call the element $\pi(i)$ of the permutation a fixed element if $\pi(i) = i$. Moreover, let the home position of an element k be position

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