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Free polygon enumeration and the area of an integral polygon ☆

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Abstract

We introduce the notion of *free polygons* as combinatorial building blocks for convex integral polygons; that is, polygons with vertices having integer coordinates. In this context, an Euler-type formula is derived for the number of integer points in the interior of an integral polygon. This leads in turn to a formula for the area of an integral polygon P via the enumeration of free integral triangles and parallelograms contained inside P. © 2000 Elsevier Science B.V. All rights reserved.

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It is well known that if a convex polygon P in the Euclidean plane is triangulated using f_0 vertices, f_1 edges, and f_2 triangles, then *Euler's formula*

$$f_0 - f_1 + f_2 = 1 \tag{1}$$

holds independently of which triangulation we chose for P. When applied to *integral polygons* (polygons with integer point vertices), variations on Euler's formula lead to a formula for the number of vertices contained in the interior of a polygon. For integral polygons one also obtains Pick's theorem, a formula for the area of an integral polygon first proved at least 100 years ago [15] and since generalized by Reeve [16,17], Macdonald [11], Ehrhart [2], Hadwiger and Wills [6], and many others [3–5].

While formula (1) holds independently of which triangulation we chose for P, the input data for each instance of (1) does depend on some initial triangulation; that is, expression of the Euler formula (1) requires the numbers f_0, f_1, f_2 for some choice of triangulation of P.

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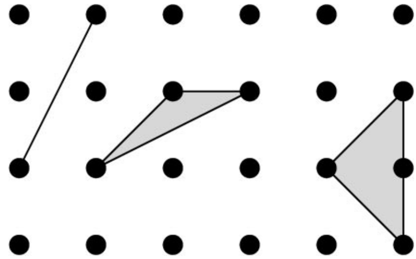

Fig. 1. A free edge, a free triangle, and a non-free triangle (from left to right).

In the present work we consider the entire family of edges, triangles, and closed convex integral polygons Q contained inside an integral polygon P, such that each convex polygon Q is *free*. This means that the vertices (extreme points) of Q must also be integral points, while Q should contain no integral points that are not extreme points. See Fig. 1. These *free polygons* are the building blocks of any convex cell decomposition of P in which the 0-cells are precisely the family of integer points $P \cap \mathbb{Z}^2$. We are, in some sense, considering *every triangulation of P at once* with the single restriction that we use the points $P \cap \mathbb{Z}^2$ as vertices. It turns out that analogues of many classical Euler-type formulas hold in this context.

In forthcoming articles [7,8] the author will present a general Euler-type relation for *valuations* on a locally finite family of polytopes in Euclidean space of arbitrary finite dimension, working in the context of free polytope enumeration. While the general theorem requires substantial machinery to prove, the case of valuations on polygons can be dealt with using a straightforward combinatorial approach. These examples provide insight into some of the most fundamental polygon functionals from an enumerative perspective not previously considered.

In the present self-contained note we consider this new perspective on Euler's classical formula using purely combinatorial techniques. Instead of considering a fixed triangulation of P, and then counting the edges, faces, etc., we will count the number of free integral polygons Q of each free type contained inside P, independently of any particular convex cell decomposition. These parameters will be found to satisfy an analogue of Euler's classical formula (Proposition 2.2), leading in turn to a formula for the number of integer points in the interior of an integral polygon (Theorem 3.1). In analogy to Pick's theorem (Theorem 4.1), we also derive a formula for the area of an integral polygon P via the enumeration of *free* integral triangles and parallelograms contained inside P (Theorem 4.2).

1. Preliminaries

A *convex polygon* is a bounded intersection of a finite collection of half-planes in $\mathbb{R}^2$ (or, alternatively, the convex hull of a finite set of points in $\mathbb{R}^2$).

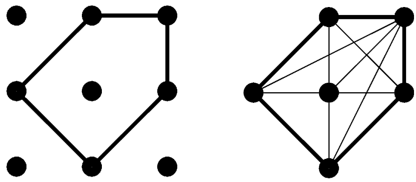

Fig. 2. An integral polygon and the free polygons contained inside it.

Let $\mathbb{Z}^2$ denote the set of all points in $\mathbb{R}^2$ having integer coordinates. A convex polygon P is an *integral polygon* if all of the vertices (extreme points) of P are points of $\mathbb{Z}^2$. Denote by $\mathscr{I}^2$ the set of all convex integral polygons.

A polygon $P \in \mathscr{I}^2$ is said to be *free* if $P \cap \mathbb{Z}^2$ consists only of the extreme points of P. In other words, P is free if no integral point of P lies in the convex hull of any of the others. For example, a free edge and a free triangle are exhibited in Fig. 1. The triangle at the right of Fig. 1 is not free, since its vertical edge contains an integer point that is not an extreme point of the triangle. The pentagonal region in Fig. 2 also is *not* free. Note that $P \in \mathscr{I}^2$ is free if and only if every $Q \in \mathscr{I}^2$ contained in P is also free. The following proposition classifies all free integral polygons.

Proposition 1.1. *Suppose $P \in \mathscr{I}^2$ is non-empty. If P is free then P is either a point, a line segment, a free triangle of area $\frac{1}{2}$, or a free parallelogram of area* 1.

In particular, free convex polygons have area $0, \frac{1}{2}$, or 1, and any free convex polygon has an affine unimodular image inside the unit square.

Proof. From the tiling properties of parallelograms (and some elementary linear algebra) it follows that a parallelogram in $\mathscr{I}^2$ is free if and only if it is the image of a translate of the unit square under a unimodular transformation. In other words, a parallelogram is free if and only if it has unit area. Since any free triangle forms a free parallelogram when pasted to its reflection through the center of an edge, an integral triangle is free if and only if has area $\frac{1}{2}$.

Let P be an arbitrary free convex polygon with at least four vertices. Any three vertices of P span a free triangle inside P. Let x_1, x_2, x_3 denote three adjacent vertices of P such that $\overline{x_1x_2}$ and $\overline{x_2x_3}$ are boundary edges of P. There exist a translation and a unimodular transformation mapping these vertices x_1, x_2, x_3 to the points $y_1 = (1,0)$, $y_2 = (0,0)$, and $y_3 = (0,1)$, respectively. Let Q denote the image of P under this transformation. Note that Q is also free, and that the edges $\overline{y_1y_2}$ and $\overline{y_2y_3}$ are boundary edges of Q. Since Q is convex and free, all remaining vertices of Q must have positive (integer) coordinates.

Let (a,b) be a fourth vertex of Q. If $a > 1$, then the triangle with vertices $(0,0)$, $(0,1)$, and (a,b) has area greater than $\frac{1}{2}$, and is consequently *not* free. This contradicts the assumption that Q is free. Therefore $a \leqslant 1$. Similarly, $b \leqslant 1$, and Q must be the unit square. It follows that P must be a free parallelogram. □

Let $\alpha_1(P) = |P \cap \mathbb{Z}^2|$; that is, $\alpha_1(P)$ gives the number of integral *points* contained in P. Similarly, define

$$\alpha_2(P) = \text{the number of free line segments contained inside } P,$$

$$\alpha_3(P) = \text{the number of free triangles contained inside } P,$$

$$\alpha_4(P) = \text{the number of free parallelograms contained inside } P.$$

Consider, for example, the integral polygon P in the left part of Fig. 2. The right part of the figure shows all free edges, triangles, and parallelograms contained in P. In this case, we have $\alpha_1(P) = 6$, $\alpha_2(P) = 13$, $\alpha_3(P) = 11$, and $\alpha_4(P) = 3$. Note that

$$\alpha_1(P) - \alpha_2(P) + \alpha_3(P) - \alpha_4(P) = 1.$$

In the next section we will show that this relation holds for arbitrary convex integral polygons.

2. Euler relations

The free polygon enumerators α_i satisfy many of the Euler relations that hold in the well-known context of simplicial triangulations.

For $P \in \mathscr{I}^2$ let $\bar{P}$ denote the collection of polygons

$$\bar{P} = \{Q \in \mathscr{I}^2 \mid Q \text{ is free and } Q \subseteq P\}.$$

For $M \subset \bar{P}$, define

$$\chi(M) = \alpha_1(M) - \alpha_2(M) + \alpha_3(M) - \alpha_4(M). \tag{2}$$

For integral polygons Q we will abuse notation by defining $\chi(Q) = \chi(\bar{Q})$.

For $P \in \mathscr{I}^2$, denote by int(P) the interior of P. When a polygon $P \in \mathscr{I}^2$ has non-empty interior, denote by ∂P the boundary of P. For $x \in P$, also define

$$P_x = \{Q \subseteq P \mid Q \text{ is free and } x \in Q\} = \{Q \in \bar{P} \mid x \in Q\}.$$

For $A \subseteq \mathbb{R}^2$, denote by $c(A)$ the convex hull of the set A.

Proposition 2.1. *Suppose $P \in \mathscr{I}^2$ has non-empty interior. For all $x \in P \cap \mathbb{Z}^2$*

$$\chi(P_x) = \begin{cases} 1 & \text{if } x \in \operatorname{int}(P), \\ 0 & \text{if } x \in \partial P. \end{cases}$$

Proof. To begin, suppose that $x \in \operatorname{int}(P) \cap \mathbb{Z}^2$. Let $e_1, \ldots, e_m$ denote the free edges in P_x, listed in clockwise order. The terms clockwise and counterclockwise are applied from a point of view 'above' the plane $\mathbb{R}^2$ (embedded in $\mathbb{R}^3$) with x as the center of rotation. See, for example, Fig. 3.

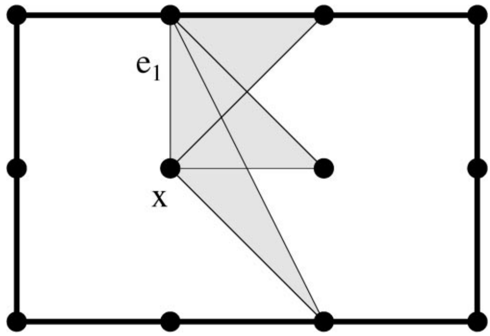

Fig. 3. The triangles in P_x having e_1 as counterclockwise edge.

Let t_1 denote the (free) triangle in P_x having edges e_1, e_2. The triangle t_1 must be free, since e_2 is the first edge of P_x to appear clockwise from e_1. Let t_2 be the *next free* triangle one can form using edges e_1 and e_i with $i > 2$. Since the triangle t_2, if it exists, is the next triangle (in clockwise order) having counterclockwise edge e_1, the convex hull $c(e_1 \cup e_2 \cup e_i)$ must be a free polygon with four vertices, a parallelogram (by Proposition 1.1), with e_2 as the diagonal through x. Call this parallelogram p_2.

Continuing, let t_3 be the next free triangle one can form with edges e_1 and e_j with $j > i$. Similarly, the existence of such a triangle t_3, implies that the convex hull $c(e_1 \cup e_i \cup e_j)$ must be a free parallelogram, with e_i as the diagonal through x. Call this parallelogram p_3, and continue this procedure until every free triangle in P_x having counterclockwise edge e_1 is accounted for.

Remark. Note that $c(e_1 \cup e_2 \cup e_j)$ is *not* a free parallelogram, since this would imply that $c(e_1 \cup e_2 \cup e_i \cup e_j)$ is a free pentagonal region, in violation of Proposition 1.1.

If we match e_1 with t_1 and match each successive t_j with the parallelogram p_j, then every triangle in P_x with e_1 as a counterclockwise edge is matched either with e_1 or with a free parallelogram in P_x sharing this edge. As was previously remarked, *all* free parallelograms in P_x with e_1 as counterclockwise edge (relative to x) are accounted for this way.

Repeating this matching procedure with each edge $e_i \in P_x$, we match every triangle $t \in P_x$ either with its counterclockwise edge or with a parallelogram p in P_x sharing its counterclockwise edge. Moreover, every edge in P_x is matched with its first clockwise triangle and every parallelogram p in P_x is matched with the triangular half of p containing both the point x and the unique diagonal of p *not* in P_x.

It follows that we have a bijective matching between the triangles of P_x and the set of all edges and parallelograms of P_x. Hence,

$$-\alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x) = 0.$$

Since the only integral *point* contained in P_x is point x, we have $\alpha_1(P_x) = 1$, so that

$$\alpha_1(P_x) - \alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x) = 1.$$

Suppose instead that $x \in \partial P \cap \mathbb{Z}^2$. The same argument applies, except that the last clockwise edge of x (a boundary edge) is never accounted for in the matching, since that last edge is the counterclockwise edge of no triangle in P_x. It follows that there is an extra -1 appearing in the sum $\alpha_1(P_x) - \alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x)$, so that

$$\alpha_1(P_x) - \alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x) = 0$$

as desired. □

We are now able to prove the free polygon analogue of the classical Euler formula (1) for a convex polygon.

Proposition 2.2 (Free Polygon Euler Formula). *If $P \in \mathscr{I}^2$ is non-empty then*

$$\chi(P) = \alpha_1(P) - \alpha_2(P) + \alpha_3(P) - \alpha_4(P) = 1. \tag{3}$$

Proof. The proposition is trivial when $\dim(P) \leqslant 1$ (i.e. when P is a point or a line segment). For the case where $\dim P = 2$ the proof is by induction on the size of $P \cap \mathbb{Z}^2$. Identity (3) clearly holds when $|P \cap \mathbb{Z}^2| \leqslant 3$. Suppose that P has non-empty interior and $|P \cap \mathbb{Z}^2| = n$. Suppose also that (3) holds for polygons containing at most $n-1$ integral points.

Let x be an extreme point (vertex) of P, and let Q denote the convex hull of the set $(P \cap \mathbb{Z}^2) - \{x\}$. Since x is a vertex of P, it follows that P is the convex hull of $Q \cup \{x\}$, and that $\bar{P} = \bar{Q} \cup P_x$ is a disjoint union. Hence, we have $\chi(P) = \chi(Q) + \chi(P_x)$.

Since $|Q \cap \mathbb{Z}^2| = n-1$, we have $\chi(Q) = 1$, by the induction assumption. Since $x \in \partial P$, Proposition 2.1 implies that $\chi(P_x) = 0$. It follows that $\chi(P) = 1$ as well. Proposition 2.2 now follows by induction. □

A generalization of Proposition 2.1 can be used to show that definition (2) of χ coincides with the classical *Euler characteristic* defined by (1) in the introduction, when extended to non-convex integral polygons. For a detailed treatment of the general case of Proposition 2.2 (in $\mathbb{R}^n$), see [7]. The classical Euler characteristic in its general form is derived in [1,9,14,20]. (See also [10,18,19] for a treatment of the Euler characteristic in combinatorial theory.)

3. Interior point enumeration

For $P \in \mathscr{I}^2$ having non-empty interior, define $I(P) = |\mathrm{int}(P) \cap \mathbb{Z}^2|$; that is, $I(P)$ counts the number of integral points in the interior of P. Denote by $B(P)$ the number of integral points contained in the boundary ∂P of P; that is, $B(P) = |\partial P \cap \mathbb{Z}^2|$. Evidently $\alpha_1(P) = I(P) + B(P)$. The functionals $I(P)$ and $B(P)$ are related to the free polygon enumerators by the following theorem.

Theorem 3.1 (Interior and boundary point enumeration). *For all convex polygons $P \in \mathscr{I}^2$ having non-empty interior,*

$$I(P) = \alpha_1(P) - 2\alpha_2(P) + 3\alpha_3(P) - 4\alpha_4(P),$$

and

$$B(P) = 2\alpha_2(P) - 3\alpha_3(P) + 4\alpha_4(P).$$

Consider, for example, the polygon P of Fig. 2, in which case $I(P)=1$ and $B(P)=5$.

Proof. By Proposition 2.1,

$$I(P) = \sum_{x \in P \cap \mathbb{Z}^2} \chi(P_x) = \sum_{x \in P \cap \mathbb{Z}^2} (\alpha_1(P_x) - \alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x)). \tag{4}$$

The sum on the right-hand side of (4) counts each free $Q \subseteq P$ once for $x \in Q \cap \mathbb{Z}^2$. That is, each free $Q \subseteq P$ is counted $\alpha_1(Q)$ times. Hence,

$$\sum_{x \in P \cap \mathbb{Z}^2} (\alpha_1(P_x) - \alpha_2(P_x) + \alpha_3(P_x) - \alpha_4(P_x)) = \sum_{\substack{Q \subseteq P \\ Q \text{ free}}} (-1)^{\alpha_1(Q)+1} \alpha_1(Q)$$

$$= \sum_{i=1}^{4} (-1)^{i+1} i \alpha_i(P).$$

The formula for $B(P)$ then follows from the fact that $\alpha_1 = I + B$. $\square$

Remark. By adjusting the sign of $I(P)$, Theorem 3.1 can be extended to include all convex integral polygons (even with empty interior). Similarly, the boundary point formula of Theorem 3.1 can be extended to a formula for the relative perimeter of any (possibly non-convex) integral polygon. For details of the more general theory, see [7].

Evidently, the values $\alpha_3(P)$ and $\alpha_4(P)$ take some considerable effort to compute, even assuming knowledge of which integral points lie inside and on the boundary of P. Some of this effort is saved if we invert the relations of Theorem 3.1 and the Free Polygon Euler Formula 2.2 to obtain the following.

Corollary 3.2. *For all convex polygons $P \in \mathscr{I}^2$ having non-empty interior,*

$$\alpha_3(P) = -3\alpha_1(P) + 2\alpha_2(P) - I(P) + 4$$

and

$$\alpha_4(P) = -2\alpha_1(P) + \alpha_2(P) - I(P) + 3.$$

Consider, for example, the polygon in Fig. 4. Here we have $\alpha_1 = 14$, $\alpha_2 = 68$, and $I = 8$. If follows that $\alpha_3 = 90$ and $\alpha_4 = 35$, as can also be seen (with some effort) by counting free polygons in Fig. 4.

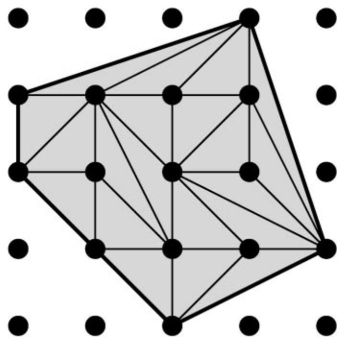

Fig. 4. A free triangulation of a convex integral polygon.

4. The area of an integral polygon

Let *Area*(P) denote the *area* of a polygon P. For *integral polygons* P, the area of P is given by the following well-known theorem of Pick [15].

Theorem 4.1 (Pick's theorem). *Suppose that P is a convex integral polygon with vertices in $\mathbb{Z}^2$. If P has non-empty interior, then*

$$\text{Area}(P) = I(P) + (1/2)B(P) - 1.$$

For example, in Fig. 4 we have $I=8$, $B=6$, and Area $=10$.

In the interest of completeness we give an elementary (though hardly original) proof of Pick's theorem. For an extensive bibliography, see [3–5].

Proof of Pick's theorem. Triangulate the convex polygon P using *all* points of $P \cap \mathbb{Z}^2$ as vertices. In particular, the points, edges, and triangles used are all *free*. See, for example, Fig. 4. Let $f_0(P)$ denote the number of vertices in the triangulation of P, $f_1(P)$ the number of edges, and $f_2(P)$ the number of triangles. The area of P is equal to the sum of the areas of the triangles used in this triangulation of P. Since these triangles are free integral triangles, each has area $\frac{1}{2}$, by Proposition 1.1. It follows that $\text{Area}(P) = (\frac{1}{2})f_2(P)$.

Let N denote the number of edge-triangle pairs (e,t) in the given triangulation such that $e \subset t$. Since each triangle t has three edges, we have $N = 3f_2(P)$. Meanwhile, each edge e belongs to *one* triangle if $e \subseteq \partial P$; otherwise e belongs to *two* triangles. (See Fig. 4.) Therefore, we have

$$3f_2(P) = N = f_1(\partial P) + 2(f_1(P) - f_1(\partial P)) = 2f_1(P) - f_1(\partial P).$$

Since P is convex with non-empty interior, the number of boundary edges of P is equal to the number of boundary vertices; that is, $f_1(\partial P)) = B(P)$. Hence, $B(P) = 2f_1(P) - 3f_2(P)$. From $f_0(P) = I(P) + B(P)$ it now follows that $I(P) = f_0(P) - 2f_1(P) + 3f_2(P)$.

After dropping the unsightly ‘(P)’, we have

$$\begin{aligned} I + (\tfrac{1}{2})B - 1 &= (f_0 - 2f_1 + 3f_2) + (\tfrac{1}{2})(2f_1 - 3f_2) - 1 \\ &= f_0 - f_1 + f_2 + (\tfrac{1}{2})f_2 - 1 \\ &= (\tfrac{1}{2})f_2 \\ &= \text{Area}(P), \end{aligned}$$

where the third equality follows from Euler’s classical formula (1). □

If we combine Pick’s theorem with the interior and boundary point formulas of Theorem 3.1 and the Euler formula of Proposition 2.2, we obtain the following area formula for integral polygons.

Theorem 4.2 (Area formula). *For all convex integral polygons P with vertices in $\mathbb{Z}^2$,*

$$\text{Area}(P) = (\tfrac{1}{2})\alpha_3(P) - \alpha_4(P).$$

For example, consider once again the polygon in Fig. 2, where $\alpha_3 = 11$, $\alpha_4 = 3$, and Area $= \frac{5}{2}$. In Fig. 4 we have $\alpha_3 = 90$, $\alpha_4 = 35$, and Area $= 10$.

In a more picturesque notation, denoting $\triangle(P) = \alpha_3(P)$ and $\square(P) = \alpha_4(P)$, we can rewrite the Area formula as follows:

$$\text{Area} = \frac{\triangle}{2} - \square. \tag{5}$$

Proof of Theorem 4.2. Once again we drop the ‘(P)’ to simplify the notation. By Theorem 3.1 and Pick’s Theorem 4.1,

$$\begin{aligned} \text{Area}(P) + 1 &= I + (\tfrac{1}{2})B \\ &= (\alpha_1 - 2\alpha_2 + 3\alpha_3 - 4\alpha_4) + (\tfrac{1}{2})(2\alpha_2 - 3\alpha_3 + 4\alpha_4) \\ &= \alpha_1 - \alpha_2 + (\tfrac{3}{2})\alpha_3 - 2\alpha_4 \\ &= (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) + (\tfrac{1}{2})\alpha_3 - \alpha_4 \\ &= 1 + (\tfrac{1}{2})\alpha_3 - \alpha_4, \end{aligned}$$

where the final equality follows from the free polygon Euler formula of Proposition 2.2. □

The preceding results can also be generalized to give geometric inequalities for arbitrary compact convex sets in $\mathbb{R}^2$. For a compact convex set $K \subseteq \mathbb{R}^2$, denote by $\alpha_i(K)$ the number of free integral polygons having i integer vertices and contained inside K. A *lattice rectangle* refers to a rectangle having vertices in $\mathbb{Z}^2$ and boundary edges parallel to the coordinate axes.

Corollary 4.3. *Suppose a compact convex set $K \subseteq \mathbb{R}^2$ has non-empty interior. Then*

$$\text{Area}(K) \geqslant (\tfrac{1}{2})\alpha_3(K) - \alpha_4(K)$$

with equality if and only if K is an integral polygon. Moreover,

$$\text{Perimeter}(K) \geqslant 2\alpha_2(K) - 3\alpha_3(K) + 4\alpha_4(K)$$

with equality if and only if K is a lattice rectangle.

Compare Corollary 4.3 to Nosarzewska's inequality [3], for example.

Proof. Suppose that K is a compact convex set in $\mathbb{R}^2$ with non-empty interior, and let P be the convex hull of $K \cap \mathbb{Z}^2$. We then have $\text{Area}(K) \geqslant \text{Area}(P)$, while $\alpha_i(K) = \alpha_i(P)$ for all i. By Theorem 4.2,

$$\text{Area}(K) \geqslant (\tfrac{1}{2})\alpha_3(K) - \alpha_4(K)$$

with equality if and only if $K = P$.

Similarly, $\text{Perimeter}(K) \geqslant \text{Perimeter}(P) \geqslant B(P)$, since each free boundary edge of P is at least of unit length. The perimeter inequality then follows from Theorem 3.1. In this case equality holds if and only if $K = P$ and if each free edge contained in the boundary of K has unit length. This only occurs when every boundary edge of K is parallel to a coordinate axis (horizontal or vertical); that is, if and only if K is a lattice rectangle. □

Evidently, the Euler relations and area formulas presented this article continue to apply if the vertex set $\mathbb{Z}^2$ is replaced by a linear image L. Integral points and polygons are then replaced by lattice points of L and lattice polygons, and the enumerators α_i are redefined in terms of the lattice L. The Euler relations continue to hold without change, while the formulas of Theorems 4.1 and 4.2 compute a renormalization of area in which the fundamental domain of the lattice L has unit area.

Many aspects of the theory have meaning even if $\mathbb{Z}^2$ is replaced by an arbitrary locally finite point set. Suppose that $\mathbb{A}$ is a subset of $\mathbb{R}^2$ (or even $\mathbb{R}^n$) such that $\mathbb{A} \cap B$ is a finite set for any closed Euclidean ball B in $\mathbb{R}^2$ (or $\mathbb{R}^n$). Such a set $\mathbb{A}$ is called *locally finite*. It can be shown that analogues of the Euler formulas of Section 2 and the interior point enumeration formula of Section 3 hold when the admissible vertex set $\mathbb{Z}^2$ is replaced with an arbitrary locally finite set $\mathbb{A}$. Analogues of the Area formula 4.2 can also be derived for *volume* in $\mathbb{R}^n$ as well as all *valuations* on polytopes. (An introduction to the theory of valuations in convex geometry can be found in [9]. See also [12,13] for extensive surveys.) The convexity condition can also be replaced with a much more liberal condition (triangulated manifolds with boundary). A treatment of this more general theory will be presented in the forthcoming articles [7,8].

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