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Bonnesen-type inequalities for surfaces of constant curvature

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Abstract

A Bonnesen-type inequality is a sharp isoperimetric inequality that includes an error estimate in terms of inscribed and circumscribed regions. A kinematic technique is used to prove a Bonnesen-type inequality for the Euclidean sphere (having constant Gauss curvature $\kappa > 0$) and the hyperbolic plane (having constant Gauss curvature $\kappa < 0$). These generalized inequalities each converge to the classical Bonnesen-type inequality for the Euclidean plane as $\kappa \to 0$.
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0. Introduction

A *Bonnesen-type* inequality is a sharp isoperimetric inequality that includes an error estimate in terms of inscribed and circumscribed regions. The classical example runs as follows:

Suppose that K is a compact convex set in $\mathbb{R}^2$. Denote by A_K and P_K the area and perimeter of K respectively. Let R_K denote the circumradius of K, and let r_K denote the inradius of K. Then

$$P_K^2 - 4\pi A_K \geqslant \pi^2 (R_K - r_K)^2. \tag{1}$$

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The classical isoperimetric inequality immediately follows, namely,

$$P_K^2 - 4\pi A_K \geqslant 0, \tag{2}$$

with equality if and only if $R_K = r_K$, that is, if and only if K is a Euclidean disc. Proofs of these inequalities, along with variations and generalizations, can be found in any of [1,8,9], for example.

In this note the kinematic methods of Santaló and Hadwiger are used to prove Bonnesen-type inequalities for the Euclidean sphere (having constant Gauss curvature $\kappa > 0$) and the hyperbolic plane (having constant Gauss curvature $\kappa < 0$). Section 1 outlines necessary background material from integral geometry. In Section 2 we derive the first of the two main theorems in this article, a Bonnesen-type inequality for the sphere, stated in Theorem 2.1. The second main theorem of this article, Theorem 3.1, is a Bonnesen-type inequality for the hyperbolic plane, derived in Section 3. The limiting case as $\kappa \to 0$ in either of Theorems 2.1 and 3.3 yields the classical Bonnesen inequality (1), as described above. A brief and direct proof of (1) using kinematic arguments, also described in [9], is presented at the close of Section 1 as a contrast to those of the subsequent sections.

1. Background: Integral geometry of surfaces

Denote by $\mathbb{X}_\kappa$ the surface of constant curvature κ, specifically:

$$\mathbb{X}_\kappa = \begin{cases} \text{Euclidean 2-sphere of radius } 1/\sqrt{\kappa} & \text{if } \kappa > 0, \\ \text{Euclidean plane } \mathbb{R}^2 & \text{if } \kappa = 0, \\ \text{Hyperbolic plane of constant curvature } \kappa & \text{if } \kappa < 0. \end{cases}$$

A compact set $P \subseteq \mathbb{X}_\kappa$ is a *convex polygon* if P can be expressed a finite intersection of closed half-planes (or closed hemispheres in the case of $\kappa > 0$, with the added requirement that P lie in inside an open hemisphere). A *polygon* is a finite union of convex polygons. More generally, a set $K \subseteq \mathbb{X}_\kappa$ will be called *convex* if any two points of K can be connected by a line segment inside K, where the notion of line segment is again suitably defined for each context (spherical, Euclidean, hyperbolic). For $\kappa > 0$, a convex set is again required to lie inside an open hemisphere. Denote by $\mathcal{K}(\mathbb{X}_\kappa)$ the set of all *compact* convex sets in $\mathbb{X}_\kappa$.

For $K \in \mathcal{K}(\mathbb{X}_\kappa)$, denote by A_K the *area* of K, and denote by P_K the *perimeter* of K. If $\dim K = 1$ then P_K is equal to *twice* the length of K. (This assures that perimeter P is continuous in the Hausdorff topology on compact sets in $\mathbb{X}_\kappa$.)

If K is a finite union of compact convex sets in $\mathbb{X}_\kappa$, denote by χ_K the *Euler characteristic* of K. If K is a compact convex set, then $\chi_K = 1$ whenever K is nonempty, while $\chi_\emptyset = 0$. More generally, χ extends to all finite unions of compact convex sets via iteration of the inclusion-exclusion identity:

$$\chi_{K\cup L} + \chi_{K\cap L} = \chi_K + \chi_L.$$

Our primary tool for studying inequalities will be the *principal kinematic formula* [9, p. 321] for compact convex sets in $\mathbb{X}_\kappa$.

Theorem 1.1 *(Principal Kinematic Formula for $\mathbb{X}_\kappa$). For all finite unions K and L of compact convex sets in $\mathbb{X}_\kappa$,*

$$\int_g \chi_{K\cap gL}\,dg = \chi_K A_L + \frac{1}{2\pi} P_K P_L + A_K \chi_L - \frac{\kappa}{2\pi} A_K A_L. \tag{3}$$

The integral on the left-hand side of (3) is taken with respect to area on $\mathbb{X}_\kappa$ and the invariant Haar probability measure on the group G_0 of isometries of $\mathbb{X}_\kappa$ which fix a base point $x_0 \in \mathbb{X}_\kappa$. To define this more precisely, denote by t_x the unique translation of $\mathbb{X}_\kappa$ (or minimal rotation, in the case of $\kappa > 0$) that maps x_0 to a point $x \in \mathbb{X}_\kappa$. Then define

$$\int_g \chi_{K\cap gL}\,dg = \int_{x\in\mathbb{X}_\kappa} \int_{\gamma\in G_0} \chi_{K\cap t_x(\gamma L)}\,d\gamma\,dx, \tag{4}$$

where we use the probabilistic normalization

$$\int_{\gamma\in G_0} d\gamma = 1.$$

The classical proof of Theorem 1.1 can be found in [9]. For a valuation-based proof of Theorem 1.1, see [7] (for the Euclidean and spherical cases) and [6] (for the hyperbolic plane). Surveys and other recent work on kinematic formulas in convex, integral, and Riemannian geometry and their applications include [2,5,7,9,12,13].

Choose a fixed base point $x_0 \in \mathbb{X}_\kappa$. For $r \geqslant 0$, denote by D_r the set of points in $\mathbb{X}_\kappa$ that lie at most a distance r from x_0. We will refer to D_r as the *disc of radius r* in $\mathbb{X}_\kappa$.

Recall that, for $\kappa \neq 0$,

$$P_{D_r} = \frac{2\pi}{\sqrt{\kappa}} \sin(\sqrt{\kappa} r) \quad \text{and} \quad A_{D_r} = \frac{2\pi}{\kappa}\bigl(1 - \cos(\sqrt{\kappa} r)\bigr). \tag{5}$$

See, for example, [11, p. 85]. The limiting cases as $\kappa \to 0$ yield the Euclidean formulas $P_{D_r} = 2\pi r$ and $A_{D_r} = \pi r^2$.

Theorem 1.1 leads in turn to the following version of Hadwiger's containment theorem for convex subsets of surfaces [3,4,7,9].

Theorem 1.2 *(Hadwiger's Containment Theorem). Let $K, L \in \mathcal{K}(\mathbb{X}_\kappa)$ with non-empty interiors. If*

$$P_K P_L \leqslant 2\pi(A_K + A_L) - \kappa A_K A_L, \tag{6}$$

then there exists an isometry g of $\mathbb{X}_\kappa$ such that either $gK \subseteq L$ or $gL \subseteq K$.

Proof. First, consider the case in which K and L are convex *polygons* in $\mathbb{X}_\kappa$. Suppose that, for every isometry g, we have $gK \nsubseteq \operatorname{int}(L)$ and $gL \nsubseteq \operatorname{int}(K)$. In this instance, whenever the K and

gL overlap, the boundary intersection ∂K and ∂L will consist of a discrete set of 2 or more points (except for a measure zero set of motions g). In other words, for almost all isometries g,

$$\chi_{\partial K \cap g\partial L} \geqslant 2\chi_{K \cap gL}.$$

On integrating both sides with respect to g, it follows from the kinematic formula (3) that

$$\frac{1}{2\pi} P_{\partial K} P_{\partial L} \geqslant 2\left(A_K + \frac{1}{2\pi} P_K P_L + A_L - \frac{\kappa}{2\pi} A_K A_L\right).$$

Recall that $P_{\partial K} = 2P_K$, and similarly $P_{\partial L} = 2P_L$, so that

$$\frac{1}{\pi} P_K P_L \geqslant A_K + \frac{1}{2\pi} P_K P_L + A_L - \frac{\kappa}{2\pi} A_K A_L.$$

Hence,

$$P_K P_L \geqslant 2\pi(A_K + A_L) - \kappa A_K A_L.$$

In other words, if (6) holds with *strict* inequality ($<$) then there exists an isometry g such that either $gK \subseteq \operatorname{int}(L)$ or $gL \subseteq \operatorname{int}(K)$. Since the set $\{(K, L, g) \mid gK \subseteq L \text{ or } gL \subseteq K\}$ is closed in the Hausdorff topology, the theorem also holds for the case of equality in (6), as well as for compact convex sets K and L in $\mathbb{X}_\kappa$ that are not polygons. □

We will also make use of the following elementary fact about the perimeter of compact convex subsets of $\mathbb{X}_\kappa$.

Proposition 1.3. *Suppose that $K, L \in \mathcal{K}(\mathbb{X}_\kappa)$ and suppose that $K \subseteq L$. Then $P_K \leqslant P_L$.*

In other words, perimeter is monotonic on compact convex sets in $\mathbb{X}_\kappa$.

Evidently Proposition 1.3 is not true for arbitrary (non-convex) sets. Nor does it hold for convex-like subsets of the sphere that do not lie inside an open hemisphere.

Proof. Suppose that $\kappa \leqslant 0$ (so that we consider the Euclidean or hyperbolic plane). Let N be a line segment of length d, so that $P_N = 2d$. Note that $\chi_{N \cap K} = 1$ if and only if $N \cap K \neq \emptyset$; otherwise $\chi_{N \cap K} = 0$. Moreover, $\chi_{N \cap K} \leqslant \chi_{N \cap L}$ since $K \subseteq L$. On averaging over all motions of N, the kinematic formula, Theorem 1.1, implies that

$$A_K + \frac{1}{2\pi} P_N P_K \leqslant A_L + \frac{1}{2\pi} P_N P_L,$$

so that

$$\frac{A_K}{2d} + \frac{1}{2\pi} P_K \leqslant \frac{A_L}{2d} + \frac{1}{2\pi} P_L,$$

for all $d > 0$. Taking the limit as $d \to \infty$ yields $P_K \leqslant P_L$.

For $\kappa > 0$ (the Euclidean sphere) replace the line segment N with a great circle C. Recall that convex sets in the sphere are required to lie inside an open hemisphere, so that $\chi_{C \cap K} = 1$ if and only if $C \cap K \neq \emptyset$, and $\chi_{C\cap K} \leqslant \chi_{C\cap L}$ whenever $K \subseteq L$. Because $\chi_C = A_C = 0$, the kinematic formula of Theorem 1.1 implies that

$$\frac{1}{2\pi} P_C P_K \leqslant \frac{1}{2\pi} P_C P_L,$$

so that $P_K \leqslant P_L$ once again. □

After setting $\kappa = 0$ in Theorem 1.2, a kinematic proof of the classical Bonnesen-type inequality (1) is straightforward.

Proof of the inequality (1). If K is a disc, then both sides of the inequality (1) are equal to zero.

Suppose that $K \in \mathcal{K}(\mathbb{R}^2)$ is not a disc, so that $r_K < R_K$. For $r_K < \epsilon < R_K$ we can apply Theorem 1.2 to K and the disc D_ϵ to obtain

$$P_K P_{D_\epsilon} > 2\pi(A_K + A_{D_\epsilon}).$$

It follows that

$$P_K 2\pi\epsilon > 2\pi\left(A_K + \pi\epsilon^2\right).$$

In other words,

$$f(\epsilon) = -\pi\epsilon^2 + \epsilon P_K - A_K > 0,$$

for all $\epsilon \in (r_K, R_K)$. Since the leading coefficient of the quadratic polynomial $f(\epsilon)$ is negative, it follows that f has two distinct roots, separated by the interval (r_K, R_K).

Hence, $\delta(f)/\pi^2 \geqslant (R_K - r_K)^2$, where $\delta(f)$ is the discriminant of f. In other words,

$$P_K^2 - 4\pi A_K \geqslant \pi^2(R_K - r_K)^2. \qquad \square$$

2. Isoperimetry in $\mathbb{S}^2$

In this section we consider the case $\mathbb{X}_\kappa$ for $\kappa > 0$. For simplicity of notation, we first consider the case $\kappa = 1$. A restatement of the main results for general constant curvature $\kappa > 0$ is then given at the end of the section; the proofs are entirely analogous to the case $\kappa = 1$.

Denote by $\mathbb{S}^2$ the Euclidean unit sphere in $\mathbb{R}^2$. For $K \in \mathcal{K}(\mathbb{S}^2)$ define the *circumradius* R_K to be the greatest lower bound of all radii R such that some spherical disc of radius R contains K. Similarly, define the *inradius* r_K to be the least upper bound of all radii r such that K contains a spherical disc (i.e. spherical cap) of radius r. Evidently $r_K \leqslant R_K$, with equality if and only if K is a spherical disc. Our restriction that a convex set must always lie in an open hemisphere implies that $R_K < \frac{\pi}{2}$.

We will use Theorem 1.2 to prove the following Bonnesen-type inequality for the sphere $\mathbb{S}^2$.

Theorem 2.1 *(Bonnesen-Type Inequality for $\mathbb{S}^2$). Suppose $K \in \mathcal{K}(\mathbb{S}^2)$. Then*

$$P_K^2 - A_K(4\pi - A_K) \geqslant \frac{(\sin R_K - \sin r_K)^2((2\pi - A_K)^2 + P_K^2)^2}{4(2\pi - A_K)^2}. \tag{7}$$

The inequality (7) has the following simplification that also provides equality conditions.

Corollary 2.2 *(Simplified Bonnesen-Type Inequality for $\mathbb{S}^2$). Suppose $K \in \mathcal{K}(\mathbb{S}^2)$. Then*

$$P_K^2 - A_K(4\pi - A_K) \geqslant \frac{1}{4}(\sin R_K - \sin r_K)^2(2\pi - A_K)^2 \tag{8}$$

with equality if and only if K is a spherical disc.

Proof of Corollary 2.2. Since $P_K^2 \geqslant 0$,

$$\frac{(\sin R_K - \sin r_K)^2((2\pi - A_K)^2 + P_K^2)^2}{4(2\pi - A_K)^2} \geqslant \frac{1}{4}(\sin R_K - \sin r_K)^2(2\pi - A_K)^2. \tag{9}$$

The inequality (8) now follows from (7) and (9).

Equality holds in (8) and (9) if and only if either $P_K = 0$, in which case K is a single point, or if $R_K = r_K$, in which case K must be a spherical disc. □

The right-hand sides of (7) and (8) are always non-negative and are equal to zero if and only if $R_K = r_K$, that is, if and only if K is a disc. These observations yield the following classical result as an immediate corollary.

Corollary 2.3 *(Isoperimetric Inequality for $\mathbb{S}^2$). For $K \in \mathcal{K}(\mathbb{S}^2)$,*

$$P_K^2 \geqslant A_K(4\pi - A_K),$$

with equality if and only if K is a spherical disc.

Note that the complement K' of K in $\mathbb{S}^2$, while not convex according to our definition, has the same boundary and perimeter as K, while the inradius and circumradius exchange roles. Meanwhile, $A_{K'} + A_K = 4\pi$, so that Corollaries 2.2 and 2.3 are transformed as follows.

Corollary 2.4 *(Alternate Simplified Bonnesen-Type Inequality for $\mathbb{S}^2$). Suppose $K \in \mathcal{K}(\mathbb{S}^2)$. Then*

$$P_K^2 - A_K A_{K'} \geqslant \frac{1}{16}(\sin R_K - \sin r_K)^2(A_K - A_{K'})^2, \tag{10}$$

so that, in particular,

$$P_K^2 - A_K A_{K'} \geqslant 0$$

with equality in both cases if and only if K is a spherical disc.

Proof. Since $A_{K'} = 4\pi - A_K$ the left-hand sides of (8) and (10) are the same. Meanwhile,

$$\begin{aligned}(A_K - A_{K'})^2 &= A_K^2 + A_{K'}^2 - 2A_K A_{K'}\\ &= A_K^2 + (4\pi - A_K)^2 - 2A_K(4\pi - A_K)\\ &= 4A_K^2 - 16\pi A_K + 16\pi^2\\ &= 4(2\pi - A_K)^2,\end{aligned}$$

so that the right-hand sides of (8) and (10) are the same as well. □

We now prove the main inequality of this section, Theorem 2.1.

Proof of Theorem 2.1. If K is a disc, then $R_K = r_K$, and both sides of (7) are equal to zero.

Suppose that $K \in \mathcal{K}(\mathbb{S}^2)$ is not a disc, so that $r_K < R_K$. For $r_K < \epsilon < R_K$ we can apply Theorem 1.2 to K and the disc D_ϵ to obtain

$$P_K P_{D_\epsilon} > 2\pi(A_K + A_{D_\epsilon}) - A_K A_{D_\epsilon}.$$

It follows from (5) that

$$\begin{aligned}P_K \sin\epsilon &> A_K + 2\pi(1-\cos\epsilon) - A_K(1-\cos\epsilon)\\ &= 2\pi(1-\cos\epsilon) + A_K\cos\epsilon.\end{aligned}$$

Setting $P = P_K$ and $A = A_K$, we have

$$P\sin\epsilon - 2\pi > (A - 2\pi)\cos\epsilon. \tag{11}$$

In order for $K \subseteq \mathbb{S}^2$ to be convex, K must be contained in a hemisphere, so that $P \leqslant 2\pi$. It follows that

$$P\sin\epsilon - 2\pi \leqslant 0,$$

so that both sides of (11) are *non-positive.* Set $x = \sin\epsilon$, so that $\cos\epsilon = \sqrt{1-x^2}$. Squaring both sides of (11) reverses the order, yielding

$$\begin{aligned}P^2x^2 - 4\pi Px + 4\pi^2 &< (A-2\pi)^2\bigl(1-x^2\bigr)\\ &= -(2\pi - A)^2x^2 + \bigl(4\pi^2 - 4\pi A + A^2\bigr),\end{aligned}$$

so that

$$f(x) = \bigl[(2\pi - A)^2 + P^2\bigr]x^2 - 4\pi Px + (4\pi - A)A < 0, \tag{12}$$

for all $x \in (\sin r_K, \sin R_K)$.

Since K is not a disc, K is not a point, so $P_K > 0$. It follows that $(2\pi - A)^2 + P^2 \geqslant P^2 > 0$, so that the quadratic polynomial $f(x)$ defined by (12) has a positive leading coefficient, and $f(x) > 0$ for sufficiently large $|x|$. Since $f(x) < 0$ for $x \in (\sin r_K, \sin R_K)$, it follows that $f(x)$ has two *real* roots, and that these two roots must lie on *different sides* of the open interval $(\sin r_K, \sin R_K)$.

The discriminant $\delta(f)$ of the quadratic polynomial f is computed as follows:

$$\begin{aligned}\delta(f) &= (4\pi P)^2 - 4\big((2\pi - A)^2 + P^2\big)(4\pi - A)A \\ &= 4\big[4\pi^2 P^2 - \big((2\pi - A)^2 + P^2\big)(4\pi - A)A\big] \\ &= 4\big[4\pi^2 P^2 - 4\pi P^2 A + P^2 A^2 - (2\pi - A)^2(4\pi - A)A\big] \\ &= 4\big[P^2\big(4\pi^2 - 4\pi A + A^2\big) - (2\pi - A)^2(4\pi - A)A\big] \\ &= 4\big[P^2(2\pi - A)^2 - (2\pi - A)^2(4\pi - A)A\big] \\ &= 4(2\pi - A)^2\big(P^2 - A(4\pi - A)\big).\end{aligned}$$

The squared distance between the roots of a quadratic polynomial f is given by its discriminant $\delta(f)$ divided by the square of its leading coefficient. Hence,

$$\frac{4(2\pi - A)^2(P^2 - A(4\pi - A))}{((2\pi - A)^2 + P^2)^2} \geqslant (\sin R_K - \sin r_K)^2 \tag{13}$$

which implies the inequality (7). □

For the general case, denote by $\mathbb{S}^2_\kappa$ the Euclidean sphere having radius $\frac{1}{\sqrt{\kappa}}$ and Gauss curvature κ. In this case our restriction that a convex set must always lie in an open hemisphere implies that $R_K < \frac{\pi}{2\sqrt{\kappa}}$. Note also that if K and K' are complements in $\mathbb{S}^2_\kappa$ then $A_K + A_{K'} = \frac{4\pi}{\kappa}$.

The inequalities of this section are now summarized in full generality. These generalized versions follow immediately from the theorems above via a scaling argument. Alternatively these more general cases can be proved in direct analogy to the proof given above for the case $\kappa = 1$.

Theorem 2.5 *(Bonnesen-Type Inequalities for $\mathbb{S}^2_\kappa$). Suppose $K \in \mathcal{K}(\mathbb{S}^2_\kappa)$, and K' denote the complement of K in $\mathbb{S}^2_\kappa$. Then the following inequalities hold*:

$$P_K^2 - A_K(4\pi - \kappa A_K) \geqslant \frac{(\sin\sqrt{\kappa}R_K - \sin\sqrt{\kappa}r_K)^2((2\pi - \kappa A_K)^2 + \kappa P_K^2)^2}{4\kappa(2\pi - \kappa A_K)^2}, \tag{14}$$

$$P_K^2 - A_K(4\pi - \kappa A_K) \geqslant \frac{1}{4\kappa}\big(\sin\sqrt{\kappa}R_K - \sin\sqrt{\kappa}r_K\big)^2(2\pi - \kappa A_K)^2, \tag{15}$$

$$P_K^2 \geqslant A_K(4\pi - \kappa A_K), \tag{16}$$

$$P_K^2 - \kappa A_K A_{K'} \geqslant \frac{\kappa}{16}\big(\sin\sqrt{\kappa}R_K - \sin\sqrt{\kappa}r_K\big)^2(A_K - A_{K'})^2. \tag{17}$$

Equality holds in (15)–(17) *if and only if K is a spherical disc.*

Note that as $\kappa \to 0^+$ the inequalities (14) and (15) of Theorem 2.5 yield the classical Bonnesen-type inequality (1) for the Euclidean plane, while (16) reduces to the classical isoperimetric inequality (2).

3. Isoperimetry in $\mathbb{H}^2$

In this section we consider the case of constant negative curvature; that is, $\kappa < 0$. To simplify notation, we first consider the case $\kappa = -1$. A restatement of the main results for general constant curvature $\kappa < 0$ is then given at the end of the section; the proofs are entirely analogous to the case $\kappa = -1$.

Let $\mathbb{H}^2$ denote the hyperbolic plane having constant negative curvature -1. For $K \in \mathcal{K}(\mathbb{H}^2)$ define the *circumradius* R_K to be the greatest lower bound of all radii R such that some hyperbolic disc of radius R contains K. Similarly, define the *inradius* r_K to be the least upper bound of all radii r such that K contains a hyperbolic disc of radius r. Evidently $r_K \leqslant R_K$, with equality if and only if K is a hyperbolic disc.

The following theorem is a limited analogue of the spherical Bonnesen-type inequality of Theorem 2.1.

Theorem 3.1. *Suppose $K \in \mathcal{K}(\mathbb{H}^2)$. If $(2\pi + A_K)^2 - P_K^2 \geqslant 0$, then*

$$P_K^2 - A_K(4\pi + A_K) \geqslant \frac{(\sinh R_K - \sinh r_K)^2((2\pi + A_K)^2 - P_K^2)^2}{4(2\pi + A_K)^2}. \tag{18}$$

The proof of Theorem 3.1 is deferred to the end of this section. Although (18) appears almost identical to the spherical Bonnesen inequality (7), up to change of sign in a few places, on more careful examination some other important differences appear.

Note that the inequality (18) may fail to hold if $(2\pi + A_K)^2 - P_K^2 < 0$. For example, if K is a line segment of length c, then $A_K = r_K = 0$, while $P_K = 2c$ and $R_K = c/2$. In this instance, the left-hand side of (18) is $O(c^2)$, while the right-hand side of (18) grows exponentially in c. This apparent deficiency will be addressed by Theorem 3.3.

Meanwhile, note that the condition $(2\pi + A_K)^2 - P_K^2 \geqslant 0$ is not as strange as it may appear, when compared carefully to the spherical case of Theorem 2.1. In the sphere we required that a convex set be contained in a hemisphere, that is, a spherical disc of radius $\frac{\pi}{2\sqrt{\kappa}}$. According to (5), this spherical disc satisfies

$$P_{D_{\frac{\pi}{2\sqrt{\kappa}}}} = \frac{2\pi}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\frac{\pi}{2\sqrt{\kappa}}\right) = \frac{2\pi}{\sqrt{\kappa}} \sin\frac{\pi}{2} = \frac{2\pi}{\sqrt{\kappa}},$$

since $\sin\frac{\pi}{2} = 1$. The next corollary involves an analogous assumption for the hyperbolic plane. In this context our replacement for the value $\pi/2$ will be $\eta = \sinh^{-1}(1) = \ln(1 + \sqrt{2})$.

Corollary 3.2. *Suppose $K \in \mathcal{K}(\mathbb{H}^2)$. If $K \subseteq D_\eta$, where $\sinh\eta = 1$, then*

$$P_K^2 - A_K(4\pi + A_K) \geqslant \frac{(\sinh R_K - \sinh r_K)^2((2\pi + A_K)^2 - P_K^2)^2}{4(2\pi + A_K)^2}.$$

Proof of Corollary 3.2. Recall from Proposition 1.3 that perimeter P_K is monotonic with respect to set inclusion when applied to *convex* sets. If $K \subseteq D_\eta$ it follows from (5) that

$$P_K \leqslant P_{D_\eta} = \frac{2\pi}{i}\sin(i\eta) = 2\pi\sinh\eta = 2\pi,$$

where $i^2 = -1$. It follows that

$$(2\pi + A_K)^2 - P_K^2 \geqslant 4\pi^2 + 4\pi A_K + A_K^2 - 4\pi^2 = 4\pi A_K + A_K^2 \geqslant 0,$$

so that Theorem 3.1 applies. □

If, contrary to the hypothesis of Theorem 3.1, we have $(2\pi + A_K)^2 - P_K^2 < 0$, then

$$P_K^2 - A_K(4\pi + A_K) > (2\pi + A_K)^2 - A_K(4\pi + A_K) = 4\pi^2. \tag{19}$$

Combining (19) with Theorem 3.1 yields the following theorem.

Theorem 3.3. *Suppose $K \in \mathcal{K}(\mathbb{H}^2)$. Then*

$$P_K^2 - A_K(4\pi + A_K) \geqslant \min\left(4\pi^2, \frac{(\sinh R_K - \sinh r_K)^2((2\pi + A_K)^2 - P_K^2)^2}{4(2\pi + A_K)^2}\right). \tag{20}$$

The right-hand side of (20) is always non-negative and is equal to zero if and only if $R_K = r_K$, that is, if and only if K is a disc. These observations yield the following corollary.

Corollary 3.4 *(Isoperimetric Inequality for $\mathbb{H}^2$). For $K \in \mathcal{K}(\mathbb{H}^2)$,*

$$P_K^2 \geqslant A_K(4\pi + A_K).$$

Equality holds if and only if K is a hyperbolic disc.

Proof of Theorem 3.1. If K is a disc, then $R_K = r_K$ and both sides of (18) are equal to zero.

Suppose that $K \in \mathcal{K}(\mathbb{H}^2)$ is not a disc, so that $r_K < R_K$. For $r_K < \epsilon < R_K$ we can apply Theorem 1.2 to K and the disc D_ϵ to obtain

$$P_K P_{D_\epsilon} > 2\pi(A_K + A_{D_\epsilon}) + A_K A_{D_\epsilon}.$$

It follows from (5) that

$$P_K \sinh\epsilon > A_K + 2\pi(\cosh\epsilon - 1) + A_K(\cosh\epsilon - 1),$$

so that

$$P_K \sinh\epsilon + 2\pi > (2\pi + A_K)\cosh\epsilon. \tag{21}$$

Set $x = \sinh\epsilon$, so that $\cosh\epsilon = \sqrt{1 + x^2}$. To simplify the notation, let $P = P_K$ and $A = A_K$. Since the right-hand side of (21) is positive, we can square both sides of (21) to obtain

$$P^2x^2 + 4\pi Px + 4\pi^2 > (2\pi + A)^2(1 + x^2) = (2\pi + A)^2x^2 + (4\pi^2 + 4\pi A + A^2),$$

so that

$$f(x) = \left[P^2 - (2\pi + A)^2\right]x^2 + 4\pi Px - (4\pi + A)A > 0, \tag{22}$$

for all $x \in (\sinh r_K, \sinh R_K)$.

Recall that, by the hypothesis of the theorem, $P^2 \leqslant (2\pi + A)^2$.

If $P^2 < (2\pi + A)^2$, then the quadratic polynomial $f(x)$ defined by (22) has a negative leading coefficient, so that $f(x) < 0$ for sufficiently large $|x|$. Since $f(x) > 0$ for $x \in (\sinh r_K, \sinh R_K)$, it follows that $f(x)$ has two *real* roots, and that these two roots must lie on *different sides* of the open interval $(\sinh r_K, \sinh R_K)$.

The discriminant $\delta(f)$ of f is computed as follows:

$$\begin{aligned}
\delta(f) &= (4\pi P)^2 + 4\big(P^2 - (2\pi + A)^2\big)(4\pi + A)A \\
&= 4\big[4\pi^2 P^2 + \big(P^2 - (2\pi + A)^2\big)(4\pi + A)A\big] \\
&= 4\big[4\pi^2 P^2 + 4\pi P^2 A + P^2 A^2 - (2\pi + A)^2(4\pi + A)A\big] \\
&= 4\big[P^2\big(4\pi^2 + 4\pi A + A^2\big) - (2\pi + A)^2(4\pi + A)A\big] \\
&= 4\big[P^2(2\pi + A)^2 - (2\pi + A)^2(4\pi + A)A\big] \\
&= 4(2\pi + A)^2\big(P^2 - A(4\pi + A)\big).
\end{aligned}$$

The squared distance between the roots of a quadratic polynomial f is given by its discriminant $\delta(f)$ divided by the square of its leading coefficient. Therefore,

$$\frac{4(2\pi + A)^2(P^2 - A(4\pi + A))}{((2\pi + A)^2 - P^2)^2} \geqslant (\sinh R_K - \sinh r_K)^2,$$

from which (18) then follows.

Finally, if $P^2 = (2\pi + A)^2$, then the right-hand side of (18) is zero, while the left-hand side reduces to the positive value $4\pi^2$. □

For the general case of constant negative curvature $\kappa < 0$, let $\lambda = |\kappa|$, and let $\mathbb{H}^2_\lambda$ denote the hyperbolic plane having Gauss curvature $-\lambda$.

The inequalities of this section are now summarized in full generality. These generalized versions follow immediately from the theorems above via a scaling argument. Alternatively these more general cases can be proved in direct analogy to the proof given above for the case $\kappa = -1$ (that is, $\lambda = 1$).

Theorem 3.5. *Suppose $K \in \mathcal{K}(\mathbb{H}^2_\lambda)$. If $(2\pi + \lambda A_K)^2 - \lambda P_K^2 \geqslant 0$, then*

$$P_K^2 - A_K(4\pi + \lambda A_K) \geqslant \frac{(\sinh\sqrt{\lambda}R_K - \sinh\sqrt{\lambda}r_K)^2((2\pi + \lambda A_K)^2 - \lambda P_K^2)^2}{4\lambda(2\pi + \lambda A_K)^2}. \tag{23}$$

More generally, if $K \in \mathcal{K}(\mathbb{H}^2_\lambda)$ then

$$P_K^2 - A_K(4\pi + \lambda A_K) \geqslant \min\left(\frac{4\pi^2}{\lambda}, \frac{(\sinh\sqrt{\lambda}R_K - \sinh\sqrt{\lambda}r_K)^2((2\pi + \lambda A_K)^2 - \lambda P_K^2)^2}{4\lambda(2\pi + \lambda A_K)^2}\right).$$

In particular, if $K \subseteq D_{\frac{\eta}{\sqrt{\lambda}}}$, where $\eta = \ln(1+\sqrt{2})$; i.e., where $\sinh\eta = 1$, then the inequality (23) holds. More generally, for all $K \in \mathcal{K}(\mathbb{H}^2_\lambda)$,

$$P_K^2 \geqslant A_K(4\pi + \lambda A_K), \tag{24}$$

where equality holds if and only if K is a hyperbolic disc.

In analogy to the spherical case, if we let $\kappa \to 0^-$, so that $\lambda \to 0^+$, then the inequalities of Theorem 3.5 yield the classical Bonnesen-type inequality (1) for the Euclidean plane, while (24) reduces to the classical isoperimetric inequality (2).

The kinematic approach to isoperimetric inequalities also leads to generalizations of (1) for the mixed area $A(K, L)$ of compact convex sets in $\mathbb{R}^2$ [9,10]. This mixed area arises in the computation of the area of the Minkowski sum $K + L$, which is itself a convolution integral of functions with respect to the translative group for $\mathbb{R}^2$. Although the surfaces $\mathbb{S}^2$ and $\mathbb{H}^2$ do not admit a subgroup of isometries analogous to the translations of $\mathbb{R}^2$, it may prove worthwhile to consider convolutions over other subgroups of isometries, leading to associated kinematic formulas, containment theorems, and Bonnesen-type isoperimetric inequalities. Bonnesen-type inequalities in higher dimensions remain elusive, but perhaps recent generalizations of Hadwiger's containment theorem to dimensions greater than 2, such as those of Zhou [14,15], may be helpful in developing discriminant inequalities in higher dimension similar to those presented in this article.

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