A SHORT PROOF OF HADWIGER'S CHARACTERIZATION THEOREM

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Abstract. One of the most beautiful and important results in geometric convexity is Hadwiger's characterization theorem for the quermassintegrals. Hadwiger's theorem classifies *all* continuous rigid motion invariant valuations on convex bodies as consisting of the linear span of the quermassintegrals (or, equivalently, of the intrinsic volumes) [**4**]. Hadwiger's characterization leads to effortless proofs of numerous results in integral geometry, including various kinematic formulas [**7, 9**] and the mean projection formulas for convex bodies [**10**]. Hadwiger's result also provides a connection between rigid motion invariant set functions and symmetric polynomials [**1, 7**].

Unfortunately the only known proof of Hadwiger's result until now has been that given in [**4**] and is the product of a long and arduous sequence of cut and paste arguments.

The purpose of this paper is to present a new and shorter proof of Hadwiger's characterization theorem, digestible within a few minutes. En route to this result is a more general characterization of volume in Euclidean space. The proof relies almost entirely on elementary techniques, with the exception of Proposition 3.1, a well-known consequence of the theory of spherical harmonics.

§1. *Background.* Denote by $\mathscr{K}^n$ the collection of all compact convex subsets of $\mathbf{R}^n$, that is, n-dimensional Euclidean space. The elements of $\mathscr{K}^n$ are also known as *convex bodies.* A convex body K is *centred about the origin*, if K is symmetric under reflection through the origin; that is, if $K=-K$. A convex body K is *centred*, if there exists a translate of K that is centred about the origin. Denote by $\mathscr{K}_c^n$ the collection of all centred convex bodies in $\mathscr{K}^n$.

A convex body $K \in \mathscr{K}^n$ is determined uniquely by its *support function*, h_K: $\mathbf{S}^{n-1} \to \mathbf{R}$, defined by $h_K(u) = \max_{x \in K} \{x \,.\, u\}$, where . denotes the standard inner product on $\mathbf{R}^n$.

For all $K, L \in \mathscr{K}^n$ and all $\lambda \geqslant 0$, the *Minkowski sum* $K + \lambda L$ is defined by

$$K + \lambda L = \{x + \lambda y : x \in K \text{ and } y \in L\},$$

and has support function $h_{K+\lambda L} = h_K + \lambda h_L$.

Every convex body K has a volume, denoted $V(K)$. Let B denote the unit ball in $\mathbf{R}^n$, centred at the origin. For all $\varepsilon \geqslant 0$, the volume $V(K+\varepsilon B)$ is given by *Steiner's formula* (see [**10**]):

$$V(K+\varepsilon B) = W_0(K) + \binom{n}{1} W_1(K)\varepsilon + \binom{n}{2} W_2(K)\varepsilon^2 + \ldots + W_n(K)\varepsilon^n.$$

[MATHEMATIKA, **42** (1995), 329–339]

For $0 \leqslant i \leqslant n$, the coefficient $W_i(K)$ depends only on the body K (independent of ε) and is called *i-th quermassintegral* of K. Setting $\varepsilon=0$, we see that

$$V(K)=W_0(K).$$

The definition for W_i depends on the dimension n of the ambient Euclidean space. The resultant ambiguity is eliminated by replacing the quermassintegrals with McMullen's *intrinsic volumes* [**5**][**10**, p. 210], defined by

$$V_i(K)=\binom{n}{i}\frac{W_{n-i}(K)}{\kappa_{n-i}},$$

for $0 \leqslant i \leqslant n$, where κ_{n-i} denotes the $(n-i)$-volume of the $(n-i)$-dimensional unit ball. The intrinsic volumes are normalized so that each V_i is equal to the i-dimensional volume when restricted to an i-dimensional subspace of $\mathbf{R}^n$.

A sequence of convex bodies K_i is said to *converge* to K in the *Hausdorff topology*, if, for all $\varepsilon>0$, there exists $N>0$ such that

$$K_i \subseteq K+\varepsilon B \qquad \text{and} \qquad K \subseteq K_i+\varepsilon B$$

for all $i>N$. In this case we write $K_i \to K$.

A function $\mu: \mathscr{K}^n \to \mathbf{R}$ is called a *valuation* on $\mathscr{K}^n$ if $\mu(\varnothing)=0$, where $\varnothing$ is the empty set, and

$$\mu(K \cup L)=\mu(K)+\mu(L)-\mu(K \cap L), \tag{1}$$

for all $K, L \in \mathscr{K}^n$ such that $K \cup L \in \mathscr{K}^n$ as well.

A valuation μ on $\mathscr{K}^n$ is said to be *simple*, if μ vanishes on all convex bodies of dimension strictly less than n.

A valuation μ on $\mathscr{K}^n$ is said to be *continuous* if, for any convergent sequence $K_i \to K$ in $\mathscr{K}^n$,

$$\lim_{i\to\infty} \mu(K_i)=\mu(K).$$

The condition on (1) that $K \cup L$ be convex may seem excessively restrictive. However, any continuous valuation μ on $\mathscr{K}^n$ can be extended in a unique way to the lattice Polycon $(\mathbf{R}^n)$ of *polyconvex* subsets of $\mathbf{R}^n$; that is, the set of all *finite unions* of compact convex subsets of $\mathbf{R}^n$. The extension is constructed as follows. Given a valuation μ on $\mathscr{K}^n$ and a set $M \in$ Polycon $(\mathbf{R}^n)$, express M as a finite union of convex bodies,

$$M=K_1 \cup \ldots \cup K_m,$$

and then compute $\mu(K_1 \cup \ldots \cup K_m)$ by iterating (1). Groemer [**3**] has shown that this extension of μ is well-defined. In the arguments that follow, this unique extension of μ shall allow us to consider the value of μ on all finite unions of convex bodies, whether or not such unions are actually convex.

Let T_n denote the group of translations in $\mathbf{R}^n$. The group T_n is naturally isomorphic to the group $\mathbf{R}^n$ under vector addition. A valuation μ on $\mathscr{K}^n$ is *translation invariant* if $\mu(TK)=\mu(K)$ for all $T \in T_n$. Let E_n denote the group of Euclidean (or rigid) motions; that is, the group generated by all translations and rotations in $\mathbf{R}^n$. A valuation μ on $\mathscr{K}^n$ is *invariant under rigid motions* if $\mu(\varphi K)=\mu(K)$ for all $\varphi \in E_n$.

A well-known example of a continuous rigid motion invariant valuation on $\mathscr{K}^n$ is the volume V. Another example is *surface area*. It turns out that all of the intrinsic volumes $V_0, V_1, \dots, V_n$ are continuous rigid motion invariant valuations on $\mathscr{K}^n$ (see [**10**, p. 290]).

§2. *Hadwiger's characterization theorem.* Hadwiger's characterization theorem classifies all continuous valuations on $\mathscr{K}^n$ that are invariant under rigid motions. The original proof is long and difficult (see [**4**]).

THEOREM 2.1 (Hadwiger's Characterization Theorem). *Suppose that μ is a continuous rigid motion invariant valuation on $\mathscr{K}^n$. Then there exist $c_0, c_1, \dots, c_n \in \mathbf{R}$ such that, for all $K \in \mathscr{K}^n$,*

$$\mu(K) = \sum_{i=0}^{n} c_i V_i(K).$$

A valuation μ on $\mathscr{K}^n$ is said to be *homogeneous of degree i*, if

$$\mu(\alpha K) = \alpha^i \mu(K),$$

for all $\alpha \geqslant 0$ and all $K \in \mathscr{K}^n$. It is well-known that, for $0 \leqslant i \leqslant n$, the valuation V_i is homogeneous of degree i. The following is an immediate corollary of Theorem 2.1.

COROLLARY 2.2. *Suppose that μ is a continuous rigid motion invariant valuation on $\mathscr{K}^n$ that is homogeneous of degree i, where $0 \leqslant i \leqslant n$. Then there exists $c \in \mathbf{R}$ such that*

$$\mu(K) = c V_i(K),$$

for all $K \in \mathscr{K}^n$.

Theorem 2.1 and Corollary 2.2 are extremely useful for the development and proof of integral geometric and kinematic formulas for convex bodies [**1**, **6**, **7**, **9**].

To begin the proof of Hadwiger's characterization theorem, let us state the following equivalent result.

THEOREM 2.3. *Suppose that μ is a continuous rigid motion invariant simple valuation on $\mathscr{K}^n$. Then there exists $c \in \mathbf{R}$ such that $\mu(K) = cV(K)$, for all $K \in \mathscr{K}^n$.*

Proof of Equivalence. We first prove that Theorem 2.3 implies Theorem 2.1. The argument is carried out by induction on dimension. The result is obvious in dimension zero. Suppose then that Theorem 2.1 holds for the case of $\mathscr{K}^{n-1}$.

Let μ be a valuation satisfying the hypotheses of Theorem 2.1. By restricting μ to convex bodies in $\mathbf{R}^{n-1}$ we obtain a continuous rigid motion invariant valuation ν on $\mathscr{K}^{n-1}$. By the induction hypothesis, Theorem 2.1 (for dimension $n-1$) implies that there exist $c_0, \ldots, c_{n-1} \in \mathbf{R}$ such that

$$\nu(K) = \sum_{i=0}^{n-1} c_i V_i(K), \tag{2}$$

for all $K \in \mathscr{K}^{n-1}$.

Let $\eta = \mu - \sum_{i=0}^{n-1} c_i V_i$. Equation (2) implies that η vanishes on $\mathscr{K}^{n-1}$, so that the valuation η satisfies the hypothesis of Theorem 2.3. Therefore, there exists $c_n \in \mathbf{R}$, such that $\eta = c_n V = c_n V_n$ on $\mathscr{K}^n$. In other words,

$$\mu = \sum_{i=0}^{n} c_i V_i.$$

This completes the proof that Theorem 2.3 implies Theorem 2.1.

Next, assume that Theorem 2.1 holds, and suppose that μ satisfies the hypothesis of Theorem 2.3. By Theorem 2.1, there exist $c_0, c_1, \ldots, c_n \in \mathbf{R}$ such that, for all $K \in \mathscr{K}^n$,

$$\mu(K) = \sum_{i=0}^{n} c_i V_i(K).$$

Suppose that K has dimension 0. Then $V_0(K) = 1$, and $V_i(K) = 0$ for all $i > 0$. Since μ vanishes on bodies of dimension less than n, we have

$$0 = \mu(K) = \sum_{i=0}^{n} c_i V_i(K) = c_0.$$

Suppose that $c_0 = c_1 = \ldots = c_k = 0$, where $0 \leqslant k \leqslant n-2$. Let K be a convex body of dimension $k+1$ such that $V_{k+1}(K) = 1$. Then $V_i(K) = 0$ for all $i > k+1$, so that

$$0 = \mu(K) = \sum_{i=0}^{k+1} c_i V_i(K) = c_{k+1}.$$

This process can be continued until we have $c_0 = \ldots = c_{n-1} = 0$, so that

$$\mu(K) = c_n V_n(K) = c_n V(K),$$

for all $K \in \mathscr{K}^n$. This completes the proof that Theorem 2.1 implies Theorem 2.3.

So far, all of this material is well-known (see [**6**, **7**]). The purpose of this paper is to give a new and straightforward proof of Theorem 2.3.

§3. *A characterization theorem for volume.* In this section we state and prove a more general characterization theorem for volume, from which Theorem 2.3 will be seen to follow. For the sake of completeness, we recall the following facts.

Recall that a *zonotope* is a finite Minkowski sum of straight line segments. A convex body Y is called a *zonoid* if Y can be approximated in $\mathscr{K}^n$ by a convergent sequence of zonotopes [**10**, p. 183]. We shall need the following useful fact concerning zonoids and smooth convex bodies. A complete discussion of this result and its proof may be found in [2] and [10].

PROPOSITION 3.1. *Let $K \in \mathcal{K}_c^n$, and suppose that the support function h_K is C^∞. Then there exist zonoids Y_1, Y_2 such that*

$$K + Y_2 = Y_1.$$

Proof. For $g \in C^\infty(\mathbf{S}^{n-1})$, the *cosine transform* of g, denoted Cg, is given by the equation

$$Cg(u) = \int_{\mathbf{S}^{n-1}} |u \,.\, v|\, g(v)dv.$$

The transform C is a linear bijection on the space of all even C^∞ functions on $\mathbf{S}^{n-1}$. This fact is a consequence of the Funke–Hecke theorem for spherical harmonics (see [**10**, p. 184]).

Since the function $h_K\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is C^∞ and even, there exists an even C^∞ function $g\colon \mathbf{S}^{n-1} \to \mathbf{R}$ such that

$$h_K(u) = \int_{\mathbf{S}^{n-1}} |u \,.\, v| g(v)dv.$$

Let $g^+(v) = \max\{g(v), 0\}$ and let $g^-(v) = \max\{-g(v), 0\}$. Then

$$h_K(u) + \int_{\mathbf{S}^{n-1}} |u \,.\, v| g^-(v)dv = \int_{\mathbf{S}^{n-1}} |u \,.\, v| g^+(v)dv. \tag{3}$$

It is easy to check that the functions $h_{Y_1} = Cg^+$ and $h_{Y_2} = Cg^-$ each satisfy the properties of a support function of a centred convex body, which we denote by Y_1 and Y_2 respectively. Equation (3) is then equivalent to the statement that $K + Y_2 = Y_1$. Moreover, since the Riemann sums converging to the integrals in (3) are linear combinations of support functions of line segments (*i.e.*, support functions of zonotopes), it follows that Y_1 and Y_2 are zonoids.

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbf{R}^n$, and denote by $SO(n, \mathcal{B})$ the set of all rotations in $SO(n)$ that fix at least one of the $(n-2)$-dimensional subspaces spanned by $n-2$ elements of the basis $\mathcal{B}$.

PROPOSITION 3.2. *Suppose that $\varphi \in SO(n)$. Then there exists a finite collection $\varphi_1, \varphi_2, \ldots, \varphi_m \in SO(n, \mathcal{B})$ such that $\varphi = \varphi_1\varphi_2 \ldots \varphi_m$.*

Proof. The proposition holds trivially in dimension $n = 2$, since $SO(2, \mathcal{B}) = SO(2)$. Suppose that the proposition holds for dimension $n-1 \geqslant 2$.

Let $\varphi \in SO(n)$. Let $v = \varphi e_n$. Assume without loss of generality that $v \neq e_n$. Let v' denote the unit normalization of the orthogonal projection of v onto Span $\{e_1, \ldots, e_{n-1}\} = \mathbf{R}^{n-1}$. There exists $\psi \in SO(n)$ such that $\psi e_n = e_n$ and $\psi v' = e_{n-1}$. Since v lies in Span $\{v', e_n\}$, it follows that ψv lies in

Span $\{e_{n-1}, e_n\}$. Let ζ be the rotation that fixes $e_1, \ldots, e_{n-2}$ and rotates ψv to e_n. Then $\zeta \in SO(n, \mathscr{B})$, and $\zeta\psi\varphi e_n = \zeta\psi v = e_n$. Let $\eta = \zeta\psi\varphi$.

Since ψ, η both fix e_n, it follows from the induction assumption on $SO(n-1)$ that there exist $\psi_1, \ldots \psi_i, \eta_1, \ldots \eta_j \in SO(n, \mathscr{B})$ such that $\psi = \psi_1 \ldots \psi_i$ and $\eta = \eta_1 \ldots \eta_j$. Thus,

$$\varphi = \psi^{-1}\zeta^{-1}\eta = \psi_i^{-1} \ldots \psi_1^{-1}\zeta^{-1}\eta_1 \ldots \eta_j.$$

The following result provides the key step to the proof of Theorem 2.3.

THEOREM 3.3. *Suppose that μ is a continuous translation invariant simple valuation on $\mathscr{K}^n$. Suppose also that $\mu([0, 1]^n) = 0$, and that $\mu(K) = \mu(-K)$, for all $K \in \mathscr{K}^n$. Then $\mu(K) = 0$, for all $K \in \mathscr{K}^n$.*

Here $[0, 1]^n$ denotes the n-fold cartesian product of the closed unit interval $[0, 1]$ with itself; that is, a unit n-cube.

Proof. If $n = 1$ then the result follows readily, since a compact convex subset of $\mathbf{R}$ is merely a closed line segment. Since μ is simple and vanishes on the closed line segment $[0, 1]$, it must vanish on all closed line segments of rational length. It then follows from continuity that μ vanishes on all closed line segments.

For $n > 1$, assume that Theorem 3.3 holds for valuations on $\mathscr{K}^{n-1}$. Since μ is translation invariant and simple, the fact that $\mu([0, 1]^n) = 0$ implies that $\mu([0, 1/k]^n) = 0$ for all integers $k > 0$. Therefore, $\mu(C) = 0$ for every box C of rational dimensions, with sides parallel to the coordinate axes. This follows from the fact that such a box can be built up out of cubes of the form $[0, 1/k]^n$ for some $k > 0$. The continuity of μ then implies that $\mu(C) = 0$ for every box C of positive real dimensions, with sides parallel to the coordinate axes.

Next, suppose that D is a box with sides parallel to a different set of orthogonal axes. If $n = 2$ then it is easy to see that D can be cut into a finite number of pieces, *translations* of which can be pasted to form a box C with sides parallel to the original coordinate axes (see Fig. 1). Since μ is simple and translation invariant, it follows that $\mu(D) = \mu(C) = 0$. If $n > 2$, then for all rotations $\zeta \in SO(n, \mathscr{B})$, a box with sides parallel to the basis $\zeta\mathscr{B}$ can be cut, translated, and re-pasted into a box parallel to $\mathscr{B}$, using precisely the operations followed in the case $n = 2$. This works because the rotation ζ fixes at least $n-2$ of the original coordinate axes. More generally, for $\psi \in SO(n)$, Proposition 3.2 states that ψ is a finite product of elements of $SO(n, \mathscr{B})$. Therefore, a box with sides parallel to the basis $\psi\mathscr{B}$ can be cut, translated, and re-pasted into a

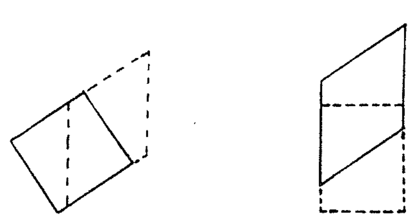

Figure 1. Re-orient a frame without use of rotations.

box parallel to $\mathscr{B}$, using a finite iteration of operations of the type used in the case $n=2$.

It follows that if D is a box with sides parallel to any frame in $\mathbf{R}^n$, then D can be transformed into a box C with sides parallel to the original coordinate axes, by means of cutting, pasting, and translations. Therefore, we have $\mu(D)=\mu(C)=0$.

Next, define a valuation τ on $\mathscr{K}^{n-1}$ as follows. Given a compact convex subset K of $\mathbf{R}^{n-1}$, set

$$\tau(K)=\mu(K\times[0,1]).$$

Note that $\tau([0,1]^{n-1})=\mu([0,1]^n)=0$. Notice also that τ satisfies the hypotheses of Theorem 3.3 in dimension $n-1$. The induction hypothesis then implies that $\tau=0$.

Since μ is simple, it then follows that $\mu(K\times[a,b])=0$, for any convex body $K\subseteq\mathbf{R}^{n-1}$ and any rational numbers a, b, with $a\leqslant b$. The continuity of μ then implies that $\mu(K\times[a,b])=0$ for all $a, b\in\mathbf{R}$. Said differently, μ is zero on any *right cylinder* with a convex base.

Let $x_1,\ldots,x_n$ be the coordinates on $\mathbf{R}^n$. We can represent $\mathbf{R}^{n-1}$ by the hyperplane $x_n=0$. The right cylinders for which we have shown μ to be zero have top and bottom that are congruent and that lie directly above and below each other. In other words, the edges connecting the top face to the bottom face are orthogonal to the hyperplane $x_n=0$.

This process can be applied to right cylinders with base in any $(n-1)$-dimensional subspace of $\mathbf{R}^n$. Since $\mu=0$ on boxes in every orientation, it follows (from the preceding argument) that $\mu=0$ on right cylinders of every orientation.

Suppose that M is a *prism*, or slanting cylinder, for which the top and bottom faces are congruent and both parallel to the hyperplane $x_n=0$, but whose cylindrical boundary is no longer orthogonal to $x_n=0$, meeting it instead at some constant angle. See Fig. 2.

Cut M into two pieces, M_1 and M_2, separated by a hyperplane that is orthogonal to the cylindrical boundary of the prism. Rearrange the pieces M_i and re-paste them together along the original (and congruent) top and bottom faces. We are then left with a right cylinder C whose surrounding boundary is orthogonal to the new top and bottom faces. Since μ remains constant under this operation, it follows that

$$\mu(M)=\mu(M_1)+\mu(M_2)=\mu(C)=0.$$

(Note. Actually, such a cut and re-arrangement is possible only if the diameter of the top/bottom of M is sufficiently small as compared to the height

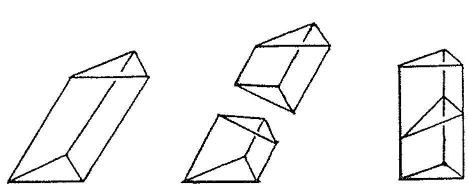

Figure 2. Turn a prism into a right cylinder.

and angle of the cylindrical boundary; *i.e.*, provided M is not too "fat". If the base of M is too large, however, we can subdivide M into "skinny" prisms by sub-dividing the top/bottom of M into convex bodies of sufficiently small diameter and considering separately each prism formed by taking the convex hull of the (disjoint) union of a piece of the bottom of M with its corresponding piece of the top of M.)

Now let P be a convex polytope having facets $P_1, \ldots, P_m$, and corresponding outward unit normal vectors $u_1, \ldots, u_m$. Let $v \in \mathbf{R}^n$, and let $\bar{v}$ denote the straight line segment connecting the point v to the origin O. Without loss of generality, let us assume that $P_1, \ldots, P_j$ are exactly those facets of P such that $u_i . v > 0$, for each $1 \leqslant i \leqslant j$. In this case, the Minkowski sum $P + \bar{v}$ can be expressed in the form

$$P + \bar{v} = P \cup \left(\bigcup_{i=1}^{j} (P_i + \bar{v}) \right),$$

where each term of the above union is either disjoint from the others, or intersects another in a convex body of dimension at most $n-1$. It follows that

$$\mu(P + \bar{v}) = \mu(P) + \left(\sum_{i=1}^{j} \mu(P_i + \bar{v}) \right).$$

Notice, however, that each term of the form $P_i + \bar{v}$ is a prism, so that $\mu(P_i + \bar{v}) = 0$. Hence,

$$\mu(P + \bar{v}) = \mu(P), \tag{4}$$

for all convex polytopes P and all line segments $\bar{v}$.

By induction over finite Minkowski sums of line segments, it immediately follows from (4) that, for all convex polytopes P and all zonotopes Z,

$$\mu(Z) = 0, \qquad \text{and} \qquad \mu(P + Z) = \mu(P).$$

The continuity of μ then implies that

$$\mu(Y) = 0, \qquad \text{and} \qquad \mu(K + Y) = \mu(K), \tag{5}$$

for all $K \in \mathscr{K}^n$ and all zonoids Y.

Next, suppose that $K \in \mathscr{K}_c^n$ has a C^∞ support function h_K. By Proposition 3.1, there exist zonoids Y_1 and Y_2 such that $K + Y_2 = Y_1$. In this case, (5) implies that

$$\mu(K) = \mu(K + Y_2) = \mu(Y_1) = 0.$$

Since any centred convex body K can be approximated by a sequence K_i of C^∞ centred convex bodies, it follows (by continuity) that μ is zero on all of $\mathscr{K}_c^n$.

Now let Δ be an n-dimensional simplex, with one vertex at the origin. Let $u_1, \ldots, u_n$ denote the other vertices of Δ, and let P be the parallelotope spanned by the vectors $u_1, \ldots, u_n$. Let $v = u_1 + \ldots + u_n$. Let ξ_1 be the hyperplane passing through the points $u_1, \ldots, u_n$, and let ξ_2 be the hyperplane passing through the points $v - u_1, \ldots, v - u_n$. Finally, denote by P_* the set of all points of P

lying between the hyperplanes ξ_1 and ξ_2. We can now write

$$P = \Delta \cup P_* \cup (-\Delta + v),$$

where each term of the union intersects another in dimension at most $n-1$. Since P and P_* are centred, we have

$$0 = \mu(P) = \mu(\Delta) + \mu(P_*) + \mu(-\Delta + v) = \mu(\Delta) + \mu(-\Delta).$$

In other words, $\mu(\Delta) = -\mu(-\Delta)$. Meanwhile, we are given that $\mu(\Delta) = \mu(-\Delta)$. Therefore $\mu(\Delta) = 0$, for any simplex Δ.

Let P be a convex polytope in $\mathbf{R}^n$. The polytope P can be expressed as a finite union of simplices

$$P = \Delta_1 \cup \ldots \cup \Delta_m,$$

such that the intersection $\Delta_i \cap \Delta_j$ has dimension less than n, for all $i \neq j$. It follows that

$$\mu(P) = \sum_{i=1}^{m} \mu(\Delta_i) = 0.$$

Since the set of all convex polytopes is dense in $\mathscr{K}^n$, the continuity of μ then implies that $\mu(K) = 0$ for all $K \in \mathscr{K}^n$.

Theorem 3.3 is equivalent to the following theorem.

THEOREM 3.4 (Volume Characterization Theorem). *Suppose that μ is a continuous translation invariant simple valuation on $\mathscr{K}^n$. Then there exists $c \in \mathbf{R}$ such that $\mu(K) + \mu(-K) = cV(K)$, for all $K \in \mathscr{K}^n$.*

Note that Theorem 3.4 implies that $\mu(K) = (c/2)V(K)$ for all centred convex bodies $K \in \mathscr{K}_c^n$.

Proof of equivalence. Suppose that μ is a continuous translation invariant simple valuation on $\mathscr{K}^n$. For $K \in \mathscr{K}^n$, define

$$\nu(K) = \mu(K) + \mu(-K) - 2\mu([0,1]^n)V(K).$$

Then ν satisfies the hypotheses of Theorem 3.3, so that $\nu(K) = 0$ for all $K \in \mathscr{K}^n$. Therefore,

$$\mu(K) + \mu(-K) = cV(K),$$

where $c = 2\mu([0,1]^n)$. Hence, Theorem 3.3 implies Theorem 3.4. The reverse implication is obvious.

§4. *Proof of Hadwiger's characterization theorem.* We saw in Section 2 that Hadwiger's characterization theorem (Theorem 2.1) is equivalent to Theorem 2.3. The following fact is required for the proof of Theorem 2.3. This proposition and its proof have been taken almost directly from [**8**, pp. 16–17].

PROPOSITION 4.1. *Let Δ be an n-dimensional simplex. There exist polytopes $P_1, \ldots, P_m$ such that*

$$\Delta = P_1 \cup \ldots \cup P_m,$$

where each term of this union intersects another in dimension at most $n-1$, and where each of the polytopes P_i is symmetric under a reflection across a hyperplane.

Proof. Let $x_0, \ldots, x_n$ be the vertices of Δ, and let Δ_i be the facet of Δ opposite to x_i. Let z be the centre of the inscribed sphere of Δ, and let z_i be the foot of the perpendicular from z to the facet Δ_i. For all $i<j$, let $A_{i,j}$ denote the convex hull of z, z_i, z_j, and the face $\Delta_i \cap \Delta_j$. Then

$$\Delta = \bigcup_{0 \leqslant i < j \leqslant n} A_{i,j},$$

where the distinct terms $A_{i,j}$ of this union intersect in at most dimension $n-1$. It is also evident that each $A_{i,j}$ is symmetric under reflection across the $n-1$ hyperplane determined by the point z and the face $\Delta_i \cap \Delta_j$. Now re-label the polytopes $A_{i,j}$ by a linear ordering $P_1, \ldots P_m$, where $m = \frac{1}{2}n(n+1)$. This gives

$$\Delta = P_1 \cup \ldots \cup P_m,$$

where the polytopes P_i satisfy the desired conditions.

We now re-state and prove Theorem 2.3.

THEOREM 4.2. *Suppose that μ is a continuous rigid motion invariant simple valuation on $\mathscr{K}^n$. Then there exists $c \in \mathbf{R}$ such that $\mu(K) = cV(K)$, for all $K \in \mathscr{K}^n$.*

Proof. Since μ is translation invariant (as well as rotation invariant) and simple, Theorem 3.4 implies the existence of $a \in \mathbf{R}$ such that $\mu(K) + \mu(-K) = aV(K)$, for all $K \in \mathscr{K}^n$.

Let Δ be a simplex in $\mathbf{R}^n$. Then we have

$$\mu(\Delta) + \mu(-\Delta) = aV(\Delta). \tag{6}$$

If the dimension n of the ambient Euclidean space is even, then Δ differs from $-\Delta$ by a rotation, so that $\mu(\Delta) = \mu(-\Delta) = (a/2)V(\Delta)$.

Meanwhile, if n is odd, by Proposition 4.1 there exist polytopes $P_1, \ldots, P_m$ such that

$$\Delta = P_1 \cup \ldots \cup P_m,$$

where each term of this union intersects another in dimension at most $n-1$, and where each of the polytopes P_i is symmetric under a reflection across a hyperplane.

It follows that each P_i differs from $-P_i$ by a rigid motion, so that $\mu(-P_i) = \mu(P_i)$. Therefore,

$$\mu(-\Delta) = \sum_{i=1}^{m} \mu(-P_i) = \sum_{i=1}^{m} \mu(P_i) = \mu(\Delta). \tag{7}$$

Taken together, (6) and (7) imply that $\mu(\Delta) = (a/2)V(\Delta)$ for any simplex Δ.

Let $c=a/2$, and suppose that P is a convex polytope in $\mathbf{R}^n$. The polytope P can be expressed as a finite union of simplices

$$P=\Delta_1\cup\ldots\cup\Delta_m,$$

such that the intersection $\Delta_i\cap\Delta_j$ has dimension less than n, for all $i\neq j$. It follows that

$$\begin{aligned}\mu(P)&=\mu(\Delta_1)+\ldots+\mu(\Delta_m)\\&=cV(\Delta_1)+\ldots+cV(\Delta_m)\\&=cV(P).\end{aligned}$$

Since the set of all convex polytopes is dense in $\mathscr{K}^n$, the continuity of μ then implies that $\mu(K)=cV(K)$ for all $K\in\mathscr{K}^n$. This concludes the proof of Theorem 4.2.

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References

1. B. Chen and G.-C. Rota. Totally invariant set functions of polynomial type. *Communications in Pure and Applied Mathematics*, 47 (1994), 187–197.
2. R. J. Gardner. *Geometric Tomography* (Cambridge University Press, 1995).
3. H. Groemer. On the extension of additive functionals on classes of convex sets. *Pacific J. Math.*, 75 (1978), 397–410.
4. H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie* (Berlin: Springer Verlag, 1957).
5. P. McMullen. Non-linear angle-sum relations for polyhedral cones and polytopes. *Math. Proc. Camb. Phil. Soc.*, 78 (1975), 247–261.
6. P. McMullen and R. Schneider. Valuations on convex bodies. In *Convexity and Its Applications*, edited by Peter M. Gruber and Jörg M. Wills (Boston: Birkhäuser Verlag, 1983).
7. G.-C. Rota. *Introduction to Geometric Probability*. Lezioni Lincee held at the Scuola Normale Superiore Pisa, Dec. 2–22, 1986.
8. C.-H. Sah. *Hilbert's Third Problem: Scissors Congruence* (San Francisco: Fearon Pitman Publishers Inc., 1979).
9. L. A. Santaló. *Integral Geometry and Geometric Probability* (Reading, MA: Addison-Wesley, 1976).
10. R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory* (Cambridge University Press, 1993).

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